Embedding and continuity envelopes of Besov-type spaces

Wen Yuan

(Joint work with Dorothee D. Haroske, Susana D. Moura, Leszek Skrzypczak and Dachun Yang)

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1. Besov-Triebel-Lizorkin type spaces built on Morrey spaces

Besov and Triebel-Lizorkin spaces

- Let $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ with

$$\begin{cases} \operatorname{supp} \widehat{\varphi}_0 \subset B(0,2), & |\widehat{\varphi}_0| > 0 \text{ on } B(0,\frac{5}{3}); \\ \operatorname{supp} \widehat{\varphi} \subset B(0,2) \setminus B(0,\frac{1}{2}), & |\widehat{\varphi}| > 0 \text{ on } B(0,\frac{5}{3}) \setminus B(0,\frac{3}{5}). \end{cases}$$

- Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$, $\varphi_j := 2^{jn}\varphi(2^j\cdot)$, $\forall\, j > 0$.

  $*$ Besov space $B^s_{p,q}(\mathbb{R}^n)$: $f \in \mathcal{S}'(\mathbb{R}^n)$ with

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \left\{ \sum_{j=0}^{\infty} 2^{jsq} \|\varphi_j * f\|^q_{L^p(\mathbb{R}^n)} \right\}^{1/q} < \infty .$$

  $*$ Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$: $f \in \mathcal{S}'(\mathbb{R}^n)$ with

$$\|f\|_{F^s_{p,q}(\mathbb{R}^n)} := \left\| \left\{ \sum_{j=0}^{\infty} 2^{jsq} |\varphi_j * f|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty .$$

Morrey spaces

- Morrey space $\mathcal{M}_{u,p}(\mathbb{R}^n)$ with $0 < p \le u \le \infty$: $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with

$$\|f\|_{\mathcal{M}_{u,p}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n, R>0} |B(x,R)|^{\frac{1}{u}-\frac{1}{p}} \left[\int_{B(x,R)} |f(y)|^p \, dy\right]^{\frac{1}{p}} < \infty .$$

  - Morrey 1938, Campanato, Brudnyi, Peetre 1960s
  - $\mathcal{M}_{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, $\mathcal{M}_{\infty,p}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$
  - $\mathcal{M}_{u,p_1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_{u,p_2}(\mathbb{R}^n)$ if $p_2 \le p_1$

Function spaces based on Morrey space

- Kozono-Yamazaki (1994), Mazzucato (2003), Tang-Xu (2005), Sawano-Tanaka (2007), Yuan-Sickel-Yang (2010), ...
- Let $s \in \mathbb{R}$, $0 < p \le u < \infty$ and $0 < q \le \infty$.
  - Besov-Morrey space $\mathcal{N}^s_{u,p,q}(\mathbb{R}^n)$: $f \in \mathcal{S}'(\mathbb{R}^n)$ with

$$\|f\|_{\mathcal{N}^s_{u,p,q}(\mathbb{R}^n)} := \left\{ \sum_{j=0}^{\infty} 2^{jsq} \|\varphi_j * f\|^q_{\mathcal{M}_{u,p}(\mathbb{R}^n)} \right\}^{1/q} < \infty.$$

  - Triebel-Lizorkin-Morrey space $\mathcal{E}^s_{u,p,q}(\mathbb{R}^n)$: $f \in \mathcal{S}'(\mathbb{R}^n)$ with

$$\|f\|_{\mathcal{E}^s_{u,p,q}(\mathbb{R}^n)} := \left\| \left\{ \sum_{j=0}^{\infty} 2^{jsq} |\varphi_j * f|^q \right\}^{1/q} \right\|_{\mathcal{M}_{u,p}(\mathbb{R}^n)} < \infty.$$

Connection to classical spaces

- $\mathcal{N}^s_{p,p,q}(\mathbb{R}^n) = B^s_{p,q}(\mathbb{R}^n)$, $\mathcal{E}^s_{p,p,q}(\mathbb{R}^n) = F^s_{p,q}(\mathbb{R}^n)$, $\mathcal{E}^0_{u,p,2}(\mathbb{R}^n) = \mathcal{M}_{u,p}(\mathbb{R}^n)$
- $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{N}^s_{u,p,q}(\mathbb{R}^n), \mathcal{E}^s_{u,p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$ is not dense in $\mathcal{N}^s_{u,p,q}(\mathbb{R}^n), \mathcal{E}^s_{u,p,q}(\mathbb{R}^n)$ when $u \neq p$
- $\mathcal{N}^s_{u,p,\min\{p,q\}}(\mathbb{R}^n) \hookrightarrow \mathcal{E}^s_{u,p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{N}^s_{u,p,\infty}(\mathbb{R}^n)$ $(B^s_{p,\min\{p,q\}}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\max\{p,q\}}(\mathbb{R}^n))$

B and F type spaces related to Q space

- [Y.-Sickel-Yang 2010] Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $p, q \in (0, \infty]$.
  - ∗ Besov-type space $B^{s,\tau}_{p,q}(\mathbb{R}^n)$: $f \in \mathcal{S}'(\mathbb{R}^n)$ with

$$\|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=\max\{j_P,0\}}^{\infty} 2^{jsq} \|\varphi_j * f\|^q_{L^p(P)} \right\}^{1/q} < \infty.$$

  - ∗ Triebel-Lizorkin-type space $F^{s,\tau}_{p,q}(\mathbb{R}^n)$: $f \in \mathcal{S}'(\mathbb{R}^n)$ with

$$\|f\|_{F^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\| \left\{ \sum_{j=\max\{j_P,0\}}^{\infty} 2^{jsq} |\varphi_j * f|^q \right\}^{1/q} \right\|_{L^p(P)} < \infty.$$

  - • $\mathcal{Q}$ — dyadic cubes in $\mathbb{R}^n$; $j_P := -\log_2 \ell(P)$ for $P \in \mathcal{Q}$
  - • [El Baraka 2002] $B^{s,\tau}_{p,q}$ for Banach cases
  - • [Yang-Y. 2008 & 2010] Homogeneous spaces $\dot{B}^{s,\tau}_{p,q}$ and $\dot{F}^{s,\tau}_{p,q}$

Motivation: relation to Q spaces

- [Essen-Janson-Peng-Xiao 2000, Dafni-Xiao 2002] The *Q space* $Q_\alpha(\mathbb{R}^n)$ with $\alpha \in \mathbb{R}$ is defined as $\{f \in L^2_{\mathrm{loc}} : \|f\|_{Q_\alpha(\mathbb{R}^n)} < \infty\}$, with

$$\|f\|_{Q_\alpha(\mathbb{R}^n)} := \sup_{I \text{ cubes}} \left\{ \frac{1}{|I|^{1-\frac{2\alpha}{n}}} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}}\, dx\, dy \right\}^{1/2} < \infty.$$

- $Q_\alpha \subset BMO$ (=BMO if $\alpha < 0$)
- [Yang-Y. 2008, 2010] $Q_\alpha = \dot{B}^{\alpha,1/2-\alpha/n}_{2,2}(\mathbb{R}^n) = \dot{F}^{\alpha,1/2-\alpha/n}_{2,2}(\mathbb{R}^n), \quad \alpha \in (0,1).$
- [Y.-Sickel-Yang 2010]

$$Q_\alpha \cap M_{\alpha/n,2}(\mathbb{R}^n) = B^{\alpha,1/2-\alpha/n}_{2,2}(\mathbb{R}^n) = F^{\alpha,1/2-\alpha/n}_{2,2}(\mathbb{R}^n), \quad \alpha \in (0,1),$$

$M_{u,p}(\mathbb{R}^n)$ is defined as $\mathcal{M}_{u,p}(\mathbb{R}^n)$ but with supremum over $R \geq 1$

Relation to known function spaces

- Relations:
  - $B^{s,0}_{p,q}(\mathbb{R}^n) = B^s_{p,q}(\mathbb{R}^n)$, $F^{s,0}_{p,q}(\mathbb{R}^n) = F^s_{p,q}(\mathbb{R}^n)$;
  - [Frazier-Jawerth 1990] $F^{s,1/p}_{p,q}(\mathbb{R}^n) = F^s_{\infty,q}(\mathbb{R}^n)$;
  - [Yang-Y. 2012] If $q \in (0,\infty)$ and $\tau \in (1/p,\infty)$, or $q = \infty$ and $\tau \in [1/p,\infty)$, then

$$B^{s,\tau}_{p,q}(\mathbb{R}^n) = B^{s+n(\tau-1/p)}_{\infty,\infty}(\mathbb{R}^n) = F^{s,\tau}_{p,q}(\mathbb{R}^n);$$

  - [Sawano-Yang-Y. 2010]

$$\mathcal{E}^s_{u,p,q}(\mathbb{R}^n) = F^{s,1/p-1/u}_{p,q}(\mathbb{R}^n); \quad \mathcal{M}_{u,p}(\mathbb{R}^n) = F^{0,1/p-1/u}_{p,2}(\mathbb{R}^n)$$

$$\mathcal{N}^s_{u,p,\infty}(\mathbb{R}^n) = B^{s,1/p-1/u}_{p,\infty}(\mathbb{R}^n), \quad \mathcal{N}^s_{u,p,q}(\mathbb{R}^n) \subsetneq B^{s,1/p-1/u}_{p,q}(\mathbb{R}^n) \quad (p < u,\ q < \infty)$$

2. Embeddings of Morrey smoothness spaces

Sobolev type embeddings for B and F spaces

Theorem (Jawerth 77, Triebel 83, Franke 86, Sickel-Triebel 95, Vybíral 08,...)

Let $s_i \in \mathbb{R}$, $p_i, q_i \in (0, \infty]$ *with* $i \in \{1, 2\}$.

(i) The embedding

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$$

holds if and only if

$$p_1 \le p_2$$

and

$$s_1 - n/p_1 > s_2 - n/p_2 \quad \text{or} \quad s_1 - n/p_1 = s_2 - n/p_2 \quad \text{and} \quad q_1 \le q_2.$$

(ii) Let $p_i \in (0, \infty)$ with $i \in \{1, 2\}$. The embedding

$$F^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow F^{s_2}_{p_2,q_2}(\mathbb{R}^n)$$

holds if and only if

$$p_1 \le p_2$$

and

$$\begin{aligned} & s_1 - n/p_1 > s_2 - n/p_2 \\ \text{or} \quad & s_1 - n/p_1 = s_2 - n/p_2 \quad \text{and} \quad p_1 < p_2 \\ \text{or} \quad & s_1 = s_2, \quad p_1 = p_2 \quad \text{and} \quad q_1 \le q_2. \end{aligned}$$

Franke-Jawerth embeddings

(iii) Let $0 < p_1 < p < p_2 \le \infty$, $q \in (0, \infty]$ and $s_1,\ s,\ s_2 \in \mathbb{R}$ with

$$s_1 - n/p_1 = s - n/p = s_2 - n/p_2.$$

The embedding

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$$

holds if and only if

$$0 < q_1 \le p \le q_2 \le \infty.$$

(iv) All above embeddings are never compact.

Embeddings for Morrey smoothness spaces

Theorem (Haroske-Skrzypczak 2012, 2013, 2014)

Let $s_i \in \mathbb{R}$, $0 < p_i \le u_i < \infty$, $q_i \in (0, \infty]$ *with* $i \in \{1, 2\}$.

(i) The embedding

$$\mathcal{N}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^n) \hookrightarrow \mathcal{N}^{s_2}_{u_2,p_2,q_2}(\mathbb{R}^n)$$

holds if and only if

$$u_1 \le u_2, \quad \frac{p_2}{u_2} \le \frac{p_1}{u_1}$$

and

$$s_1 - n/u_1 > s_2 - n/u_2 \quad \text{or} \quad s_1 - n/u_1 = s_2 - n/u_2 \quad \text{and} \quad q_1 \le q_2.$$

(ii) The embedding

$$\mathcal{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^n) \hookrightarrow \mathcal{E}^{s_2}_{u_2,p_2,q_2}(\mathbb{R}^n)$$

holds if

$$u_1 \le u_2, \quad \frac{p_2}{u_2} \le \frac{p_1}{u_1}$$

and

$$\begin{aligned} & s_1 - n/u_1 > s_2 - n/u_2 \\ \text{or} \quad & s_1 - n/u_1 = s_2 - n/u_2 \quad \text{and} \quad u_1 < u_2 \\ \text{or} \quad & s_1 = s_2, \quad u_1 = u_2 \quad \text{and} \quad q_1 \le q_2. \end{aligned}$$

Franke-Jawerth embeddings

(iii) Let $0 < p_i < u_i < \infty$ and $u_1 < u_2$ with $s_1 - n/u_1 = s_2 - n/u_2$.

(a) If $p_1 < u_1$ and $\frac{p_2}{u_2} \le \frac{p_1}{u_1}$, the embedding

$$\mathcal{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^n) \hookrightarrow \mathcal{N}^{s_2}_{u_2,p_2,q_2}(\mathbb{R}^n)$$

holds if and only if $q_2 = \infty$;

(b) If $p_1 < u_1$ and $s_1 - n/p_1 = s_2 - n/p_2$, then

$$\mathcal{N}^{s_1}_{u_1,p_1,u_2}(\mathbb{R}^n) \hookrightarrow \mathcal{E}^{s_2}_{u_2,p_2,q_2}(\mathbb{R}^n).$$

• Condition "$p_1 < u_1$" means that the left spaces do not go back to classical B or F spaces

• Main tool: wavelet decomposition

- How about embedding properties for $B^{s,\tau}_{p,q}$ and $F^{s,\tau}_{p,q}$?

- Recall that

(a) $F^{s,1/p}_{p,q}(\mathbb{R}^n) = F^s_{\infty,q}(\mathbb{R}^n)$;

(b) $B^{s,\tau}_{p,q}(\mathbb{R}^n) = B^{s+n(\tau-1/p)}_{\infty,\infty}(\mathbb{R}^n) = F^{s,\tau}_{p,q}(\mathbb{R}^n)$ when $q \in (0,\infty)$ and $\tau \in (1/p,\infty)$, or $q = \infty$ and $\tau \in [1/p,\infty)$;

(c) $\mathcal{E}^s_{u,p,q}(\mathbb{R}^n) = F^{s,1/p-1/u}_{p,q}(\mathbb{R}^n)$.

- We mainly deal with $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $\tau \in [0,1/p]$ and $q \in (0,\infty)$

3. Embedding properties for $B^{s,\tau}_{p,q}(\mathbb{R}^n)$

Wavelet basis

- To consider the embedding of Besov type spaces, one main tool is their wavelet decomposition characterization
- Let

$$\{\phi_{0,k}:\ k\in\mathbb{Z}^n\}\cup\{\psi_{i,j,k}:\ k\in\mathbb{Z}^n,\ j\in\mathbb{N}_0,\ i\in\{1,\ldots,2^n-1\}\}$$

be an orthonormal Daubechies wavelet basis with sufficient decay and moment condition.

- For $f\in B^{s,\tau}_{p,q}(\mathbb{R}^n)\subset\mathcal{S}'(\mathbb{R}^n)$, define

$$\lambda_k:=\langle f,\,\phi_{0,k}\rangle,\qquad \lambda_{i,j,k}:=\langle f,\,\psi_{i,j,k}\rangle,$$

and $\lambda(f):=\{\lambda_k\}_{k\in\mathbb{Z}^n}\cup\{\lambda_{i,j,k}\}_{i\in\{1,\ldots,2^n-1\},j\in\mathbb{N}_0,k\in\mathbb{Z}^n}$, which have been proved to make sense for certain s,p,q,τ

Sequence spaces related to $B^{s,\tau}_{p,q}$

- Let $b^{s,\tau}_{p,q}$ be the set of sequence $t = \{t_Q\}_{Q\in\mathcal{Q},\ \ell(Q)\le 1} \subset \mathbb{C}$ such that $\|t\|_{b^{s,\tau}_{p,q}} < \infty$, where

$$\|t\|_{b^{s,\tau}_{p,q}} := \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=\max(j_P,0)}^{\infty} 2^{j(s+\frac{n}{2})q} \left[\int_P \sum_{\ell(Q)=2^{-j}} |t_Q|^p \chi_Q(x)\, dx \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$

$$= \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=\max(j_P,0)}^{\infty} 2^{j(s+\frac{n}{2}-\frac{n}{p})q} \left[\sum_{\substack{\ell(Q)=2^{-j}\\ Q\subset P}} |t_Q|^p \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}}.$$

Main tool: wavelet decomposition

Theorem (Liang-Sawano-Ullrich-Yang-Y. 2013)

Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $p, q \in (0, \infty]$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in B^{s,\tau}_{p,q}$ if, and only if, $f \in \mathcal{S}'(\mathbb{R}^n)$ can be represented as

$$f = \sum_{k \in \mathbb{Z}^n} \lambda_k \phi_{0,k} + \sum_{i=1}^{2^n-1} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \lambda_{i,j,k} \psi_{i,j,k}$$

in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\|\lambda(f)\|^*_{b^{s,\tau}_{p,q}} := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{k:\ Q_{0,k} \subset P} |\lambda_k|^p \right\}^{\frac{1}{p}} + \sum_{i=1}^{2^n-1} \|\{\lambda_{i,j,k}\}_{j,k}\|_{b^{s,\tau}_{p,q}} < \infty.$$

Moreover, $\|f\|_{B^{s,\tau}_{p,q}}$ is equivalent to $\|\lambda(f)\|^*_{b^{s,\tau}_{p,q}}$.

- Form this, we know $B^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2,\tau_2}_{p_2,q_2}(\mathbb{R}^n) \Longleftrightarrow b^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow b^{s_2,\tau_2}_{p_2,q_2}(\mathbb{R}^n)$

Sobolev type embeddings for $B^{s,\tau}_{p,q}$

- Recall that $B^{s,\tau}_{p,q} = B^{s+n\tau-n/p}_{\infty,\infty}$ if $\tau > 1/p$ or $\tau = 1/p$ and $q = \infty$

Theorem (Y.-Haroske-Skrzypczak-Yang 2015)

Let $s_1, s_2 \in \mathbb{R}$, $\tau_1, \tau_2 \in [0,\infty)$ and $p_1, p_2, q_1, q_2 \in (0,\infty]$.

(i) If $\tau_2 > \frac{1}{p_2}$ or $\tau_2 = \frac{1}{p_2}$ and $q_2 = \infty$, then the embedding

$$B^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2,\tau_2}_{p_2,q_2}(\mathbb{R}^n) \ (= B^{s_2+n\tau_2-n/p_2}_{\infty,\infty}(\mathbb{R}^n))$$

holds if, and only if

$$s_1 + n\tau_1 - n/p_1 \geq s_2 + n\tau_2 - n/p_2.$$

- For Besov spaces $s_1 - n/p_1 \geq s_2 - n/p_2$

Theorem

(ii) If $\tau_1 > \frac{1}{p_1}$ *or* $\tau_1 = \frac{1}{p_1}$ *and* $q_1 = \infty$*, then the embedding*

$$(B^{s_1+n\tau_1-n/p_1}_{\infty,\infty}(\mathbb{R}^n) =)\ B^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2,\tau_2}_{p_2,q_2}(\mathbb{R}^n)$$

holds if, and only if,

$$s_1 + n\tau_1 - n/p_1 > s_2 + n\tau_2 - n/p_2 \quad \text{and} \quad \tau_2 \geq 1/p_2,$$

or

$$s_1 + n\tau_1 - n/p_1 = s_2 + n\tau_2 - n/p_2 \quad \text{and} \quad \tau_2 > 1/p_2$$
$$\text{or} \quad \tau_2 = 1/p_2 \quad \text{and} \quad q_2 = \infty\,.$$

Theorem (Y.-Haroske-Skrzypczak-Yang 2015)

Let $s_1, s_2 \in \mathbb{R}$, $\tau_1, \tau_2 \in [0, \infty)$ *and* $p_1, p_2, q_1, q_2 \in (0, \infty]$. *Assume that* $\tau_i < 1/p_i$ *or* $\tau_i = 1/p_i$ *and* $q_i \in (0, \infty)$, $i \in \{1, 2\}$. *Then*

$$B^{s_1,\,\tau_1}_{p_1,\,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2,\,\tau_2}_{p_2,\,q_2}(\mathbb{R}^n)$$

holds if **(a)** $\tau_2 - \tau_1 + 1/p_1 - 1/p_2 \geq 0, \quad \tau_1/p_2 \leq \tau_2/p_1$

and **(b)** $s_1 + n\tau_1 - n/p_1 > s_2 + n\tau_2 - n/p_2$

or $s_1 + n\tau_1 - n/p_1 = s_2 + n\tau_2 - n/p_2$

with one of the following:

$$(s_1 - s_2)(\tau_1 - \tau_2) \neq 0 : \quad \tau_1/p_2 < \tau_2/p_1;$$

$$(s_1 - s_2)(\tau_1 - \tau_2) \neq 0 : \ \tau_1/p_2 = \tau_2/p_1, \ \tau_1/q_2 \leq \tau_2/q_1;$$

$$(s_1 - s_2)(\tau_1 - \tau_2) = 0 : \quad q_1 \leq q_2.$$

(All **red conditions are also necessary***.)*

Embeddings from Besov spaces into $B^{s,\tau}_{p,q}$

Corollary

Let $s_1, s_2 \in \mathbb{R}$, $\tau_2 \in [0,\infty)$ *and* $p_1, p_2, q_1, q_2 \in (0,\infty]$. *Then*

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2,\tau_2}_{p_2,q_2}(\mathbb{R}^n)$$

holds true if and only if $\tau_2 \geq \frac{1}{p_2} - \frac{1}{p_1}$ *and*

$$s_1 - s_2 > n\tau_2 + n/p_1 - n/p_2$$

or

$$s_1 - s_2 = n\tau_2 + n/p_1 - n/p_2$$

with one of

$$\begin{cases} s_1 > s_2 \quad \text{and} \quad \tau_2 \neq 0; \\ q_1 \leq q_2 \quad \text{if} \quad s_1 = s_2 \quad \text{or} \quad \tau_2 = 0. \end{cases}$$

Embeddings from $B^{s,\tau}_{p,q}$ into Besov spaces

Corollary

Let $s_1, s_2 \in \mathbb{R}$, $\tau_1 \in [0,\infty)$ *and* $p_1, p_2, q_1, q_2 \in (0,\infty]$. *Then*

$$B^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$$

if and only if $1/p_2 \le \max\{1/p_1 - \tau_1, 0\}$, $\frac{\tau_1}{p_2} = 0$ *and*

$$s_1 - s_2 > n/p_1 - n\tau_1 - n/p_2,$$

or

$$s_1 - s_2 = n/p_1 - n\tau_1 - n/p_2$$

with one of

$$\begin{cases} p_2 = q_2 = \infty; \\ \tau_1 = 0 \quad \text{and} \quad q_1 \le q_2; \\ \tau_1 = 1/p_1,\ p_2 = \infty, \quad q_1 \le q_2, \quad \text{and} \quad q_1 < \infty. \end{cases}$$

Embeddings from $B^{s,\tau}_{p,q}$ into $C(\mathbb{R}^n)$

- $C(\mathbb{R}^n)$ — continuous and bounded functions

Corollary

Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$ *and* $p, q \in (0, \infty]$. *The embedding*

$$B^{s,\tau}_{p,q}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$$

holds if, and only if,

$$s > n\left(\frac{1}{p} - \tau\right)$$

or

$$s = n\left(\frac{1}{p} - \tau\right), \ \tau = 0 \quad \text{and} \quad q \in (0, 1] \, .$$

- $C_{\rm ub}(\mathbb{R}^n)$ — uniformly continuous and bounded functions

Corollary

Let $s \in \mathbb{R}$, $\tau \in [0,\infty)$ and $p, q \in (0,\infty]$. Then the following conditions are equivalent:

(i) $B^{s,\tau}_{p,q} \hookrightarrow C_{\rm ub}(\mathbb{R}^n)$;

(ii) $B^{s,\tau}_{p,q} \hookrightarrow C(\mathbb{R}^n)$;

(iii) $B^{s,\tau}_{p,q} \hookrightarrow L_\infty(\mathbb{R}^n)$.

Franke-Jawerth type embeddings

Theorem (Y.-Haroske-Moura-Skrzypczak-Yang 2015)

Let $s_i \in \mathbb{R}$, $p_i, q_i \in (0,\infty]$, $i \in \{1,2\}$, $\tau_1 \in [0,1/p_1)$, $\tau_2 \in [0,\infty)$, $s_1 > s_2$ and

$$s_1 - n/p_1 + n\tau_1 = s_2 - n/p_2 + n\tau_2.$$

The embeddings

$$B^{s_1,\tau_1}_{p_1,p_2}(\mathbb{R}^n) \hookrightarrow F^{s_2,\tau_2}_{p_2,q_2}(\mathbb{R}^n)$$

and

$$F^{s_1,\tau_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2,\tau_2}_{p_2,p_1}(\mathbb{R}^n)$$

hold if $\tau_1 = \tau_2$, or $\tau_1 \neq \tau_2$ and $\tau_1 p_1 < \tau_2 p_2$.
These embedding does not hold if $\tau_1 p_1 > \tau_2 p_2$.

4. Continuity envelopes in $B^{s,\tau}_{p,q}(\mathbb{R}^n)$

Continuity envelopes

- Given function f, $\omega(f,t)$ stands for the modulus of continuity:

$$\omega(f,t) := \sup_{|h|\le t} \sup_{x\in\mathbb{R}^n} |f(x+h)-f(x)|, \quad t\in(0,\infty).$$

Definition (Triebel 01, Haroske 02, 07)

Let $X \hookrightarrow C(\mathbb{R}^n)$ be some quasi-normed function space.

The continuity envelope function $\mathcal{E}_C^X : (0,\infty) \to [0,\infty]$ of X is defined by

$$\mathcal{E}_C^X(t) := \sup_{\|f\|_X\le 1} \frac{\omega(f,t)}{t}, \qquad t\in(0,\infty).$$

- $X \hookrightarrow \mathrm{Lip}^1(\mathbb{R}^n) \iff \mathcal{E}_C^X$ is bounded

Assume $X \not\hookrightarrow \mathrm{Lip}^1(\mathbb{R}^n)$. Let $\varepsilon \in (0,1)$, $H(t) := -\log \mathcal{E}_C^X(t)$, $t \in (0,\varepsilon]$, and let μ_H be the associated Borel measure. The number $u_C^X \in (0,\infty]$ is defined as the infimum of all numbers $v \in (0,\infty]$ such that

$$\left\{\int_0^{\varepsilon}\left[\frac{\omega(f,t)}{t\,\mathcal{E}_C^X(t)}\right]^v \mu_H(\mathrm{d}t)\right\}^{1/v} \le c\,\|f\|_X$$

holds for some positive constant c and all $f \in X$. The couple

$$\mathfrak{E}_C(X) := \left(\mathcal{E}_C^X(\cdot), u_C^X\right)$$

is called the continuity envelope for the function space X.

• Continuity envelopes give a precious way to describe the growth of functions in X. They have been used to study sharp and compact embeddings.

Some known results

Theorem (Triebel 01, Haroske 02)

(i) $\mathfrak{E}_C(C(\mathbb{R}^n)) = (t^{-1}, \infty)$;

(ii) for any $a \in (0, 1)$,

$$\mathfrak{E}_C(\mathrm{Lip}^a(\mathbb{R}^n)) = (t^{a-1}, \infty);$$

(iii) for $p, q \in (0, \infty]$ and $s = n/p + \sigma$ with $\sigma \in (0, 1)$,

$$\mathfrak{E}_C(B^s_{p,q}(\mathbb{R}^n)) = (t^{\sigma-1}, q), \quad \mathfrak{E}_C(F^s_{p,q}(\mathbb{R}^n)) = (t^{\sigma-1}, p);$$

(iv)

$$\mathfrak{E}_C(B^{n/p+1}_{p,q}(\mathbb{R}^n)) = (|\log t|^{1/q'}, q), \ \forall\, p \in (0, \infty],\ q \in (1, \infty];$$

$$\mathfrak{E}_C(F^{n/p+1}_{p,q}(\mathbb{R}^n)) = (|\log t|^{1/p'}, p), \ \forall\, p \in (1, \infty),\ q \in (0, \infty].$$

• Let $s \in \mathbb{R}$, $\tau \in (0, \infty)$ and $p,\ q \in (0, \infty]$. From the previous embeddings, we have

$$B^{s,\tau}_{p,q}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \Longleftrightarrow s > n(1/p - \tau)$$

and

$$B^{s,\tau}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^1(\mathbb{R}^n) \Longleftrightarrow s > n(1/p - \tau) + 1.$$

Theorem (Y.-Haroske-Moura-Skrzypczak-Yang 2015)

Let $p,\ q \in (0, \infty]$, $\tau \in (0, \infty)$ Then

$$\mathfrak{E}_C(B^{s,\tau}_{p,q}(\mathbb{R}^n)) = \begin{cases} (t^{s+n(\tau-\frac{1}{p})-1}, \infty), & \text{if} \quad n(\frac{1}{p} - \tau) < s < n(\frac{1}{p} - \tau) + 1, \\ (|\log t|, \infty), & \text{if} \quad s = n(\frac{1}{p} - \tau) + 1. \end{cases}$$

• $\mathfrak{E}_C(B^{n/p+1}_{p,q}(\mathbb{R}^n)) = (|\log t|^{1/q'}, q)$

Application I: Hardy type inequalities

Corollary (Y.-Haroske-Moura-Skrzypczak-Yang 2015)

Let $p, q \in (0, \infty]$, $\tau \in (0, \infty)$, *and* $\varepsilon \in (0, \infty)$ *be small. Let* $\varkappa$ *be an arbitrary non-negative function on* $(0, \varepsilon]$.

(i) If $n(\frac{1}{p} - \tau) < s < n(\frac{1}{p} - \tau) + 1$, *then*

$$\sup_{t \in (0,\varepsilon)} \varkappa(t)\ t^{\frac{n}{p} - n\tau - s}\, \omega(f, t) \le c\, \|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} \quad \forall f \in B^{s,\tau}_{p,q}(\mathbb{R}^n)$$

if and only if $\varkappa$ *is bounded.*

(ii) If $s = n(\frac{1}{p} - \tau) + 1$, *then*

$$\sup_{t \in (0,\varepsilon)} \varkappa(t)\, \frac{\omega(f, t)}{t\, |\log t|} \le c\, \|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} \quad \forall f \in B^{s,\tau}_{p,q}(\mathbb{R}^n)$$

if and only if $\varkappa$ *is bounded.*

Application II: approximation numbers

Approximation numbers

Let X, Y be two quasi-Banach spaces, $T \in \mathcal{L}(X, Y)$ and $k \in \mathbb{N}$. The k-th approximation number $a_k(T)$ is given by

$$a_k(T) := \inf\{\|T - S\| : S \in \mathcal{L}(X, Y),\ \mathrm{rank}\, S < k\}.$$

• Let Ω be a bounded smooth domain, e. g, $\Omega = B(0,1)$. Define $B^{s,\tau}_{p,q}(\Omega)$ via restriction.

Corollary

Let $p \in [2,\infty]$, $q \in (0,\infty]$, $\tau \in [0,\infty)$ and $n(\frac{1}{p} - \tau) < s < n(\frac{1}{p} - \tau) + 1$. Then

$$a_k \left(\mathrm{id}_\Omega : B^{s,\tau}_{p,q}(\Omega) \to C(\Omega)\right) \sim k^{-\frac{s}{n} - \tau + \frac{1}{p}}, \quad k \in \mathbb{N}.$$

• $\tau = 0$: Edmunds 1996, Caetano 1998

Thank you for your attention!