Report about the programme
Spectral Theory of Relativistic Operators
23 July - 17 August 2012
Organisers: Malcolm Brown (Cardiff), Maria J. Esteban (Paris), Karl M. Schmidt (Cardiff)
and Heinz Siedentop (Munich)

This one-month programme has been a rare, possibly unprecedented opportunity to bring together a large number of experts specialising in the area of relativistic quantum mechanics and the analysis of Dirac operators, topics which have hitherto led something of a fringe existence in the sphere of influence of the wider areas of Mathematical Physics and the Spectral Analysis of Differential Operators. The result has been a very focussed and fruitful short programme, to the great satisfaction of its participants. Most of the participants were mathematicians working in analysis, but some physicists contributed their perspective on the subject, too. Bringing experts in the field from the UK and from all over the world together led to a great pooling of expertise, but also helped to highlight the role of the UK in research leadership in the field of modern relativistic quantum theory.

One of the most thrilling and attractive topics of this programme turned out to be the mathematical study of models for the electromagnetic properties of graphene. Graphene is an atomic-scale honeycomb one-layer lattice of carbon atoms. This material has a high potential for innovative industrial applications, and the 2010 Nobel Prize in Physics was awarded to Andre Geim and Konstantin Novoselov at the University of Manchester for their "groundbreaking experiments regarding the two-dimensional material graphene". On the mathematical side, it poses the novel challenge of understanding the general behaviour of a genuinely two-dimensional, massless Dirac-type operator. In order to familiarise the programme participants with the various models for graphene current in physics and with the rationale of their derivation, the programme workshop started with a highly valued 2-hour course presented by F. Marquardt, a German physicst.

The contributions to the one-week workshop, which aimed at facilitating the exchange of information about the latest results in the important area lying at the boundary of physics and mathematics, presented recent results which can be grouped in three major themes.

1. The introductory course about the physics of graphene was complemented with a number of talks developing mathematical techniques for the study of Dirac operators and their spectra, often with direct applications to the properties of graphene.

2. Relativistic multi-particle problems pose a particularly interesting challenge. On the one hand, the Dirac equation itself does not readily yield a multi-particle theory due to its lack of lower semi-boundedness, on the other hand the multi-particle problem arises naturally and with high relevance in quantum chemistry. Among the talks on this theme, a spectacular joint result with C. Hainzl and C. Sparber was presented by Mathieu Lewin. Under the modest title "Ground state properties of graphene in Hartree-Fock theory", they were in fact able to settle an open question which had been debated in the theoretical physics literature and for which physical intuition seemed unable to give a definite answer. For instance, they explained how to construct the translation-invariant ground state for a mean-field model of graphene. Due to the exchange term, the resulting effective Fermi velocity is logarithmically divergent at zero momentum. Then they showed the existence of ground states in the presence of local defects and some properties of the associated nonlinear response.

3. The remaining workshop talks put the topics of the programme into the wider context of e.g. algebraic quantum field theory, as exemplified by Jan Derenzinski's presentation, and differential geometry, as in Dmitry Vassiliev's talk.

In addition to the workshop, the programme was enhanced through a number of seminar talks during the other weeks, which also attracted visitors from other institutions, notably the London colleges. In one of these seminars, Jan Philip Solovej presented a recent proof of a conjecture by Lieb and Wehrl, which had remained an open problem since 1978, on the classical entropy of quantum states. This result, which has now finally been proven by Lieb and Solovej, constitutes a major breakthrough in this area; it was presented publicly for the first time during our programme. A further seminar highlight, the talk by Elliot Lieb, presented inequalities for the quantum entropy. These are relevant for quantum information and have potential applications in quantum computing.

A second, more important objective of the programme, beyond the sharing of recent developments and results, was to contribute to the initiation of new research projects. The excellent conditions provided by the Isaac Newton Institute were most helpful in this respect. Indeed, throughout the month lively discussions and exchanges between the participants and visitors were taking place in the offices and around the ubiquitous blackboards. Several important projects and results have been initiated here. Due to the brevity of the one-month programme, it is too early to enumerate and assess these new projects, but several participants already mentioned concrete results they had found during these weeks. An example is work on the N-representability of the two-particle density between Bach, Knörr, Menge, and Siedentop which will have impact on both relativistic and non-relativistic multi-particle theory.

This programme was of an interdisciplinary nature, crossing the boundary between physics and mathematics, and even with some wandering in the direction of quantum chemistry. From the mathematical perspective, many results presented in the workshop were related to the question of how to locate eigenvalues of Dirac operators, how to compute them, and how to get *a priori* or *a posteriori* errors for such computations. This issue is of great practical importance in the natural sciences and their technical applications.

The overall experience has been very positive. Most of the participants told us about how happy they were to have been able to take part in the programme and how useful it had been for them to meet people who work in their area or a related field to discuss current and future research with them. The workshop also served as a window for several British scientists who had their initial training in other areas but were interested in this fast developing topic. The length of the programme seems to have fit its purpose very well, as a focussed event, with a very high level of engaged participation throughout, staking out the boundaries of an exciting and lively area of research which borders on a great variety of more established fields.

Cardiff, Paris and Munich, September 2012
Malcolm Brown, Maria J. Esteban, Karl M. Schmidt and Heinz Siedentop