Report on the Discrete Analysis Programme at INI.

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1 Aims and background

During the past decade or so there have been dramatic developments in the interaction between analysis, combinatorial number theory and theoretical computer science: in particular between harmonic analysis and combinatorial number theory and between geometric functional analysis and the theory of algorithms.

The purpose of this programme was to bring together researchers in these diverse areas of mathematics, to encourage more interaction between these fields, and to provide an opportunity for UK mathematicians to engage with an important part of the mathematical computer science community.

The programme was attended by around 125 participants in addition to those who came solely for workshops. Participants gave some 60 seminars during the programme (again in addition to workshop lectures). The "cross-fertilisation" element of the programme meant that the seminars played a more than usually important role in introducing people to one another and their research problems.

The programme was unusual in that there was a smaller programme on "Algorithms and Groups" at the Institut Henri Poincaré run at the same time, about half of which (in terms of topics) overlapped with some 20% of this one. This had some benefits in terms of travel costs for participants from the US. The INI decided to experiment with streaming the lectures of the first INI workshop to Paris so that participants there could take part interactively. While the equipment in Cambridge worked well, that in Paris was some distance from the main programme venue and the uptake by mathematicians in Paris was low. Nevertheless it seems likely that in future, scientists of all stripes will consider video-conferencing as a normal way to take part in research conferences once the necessary equipment is sufficiently widely distributed.

2 Workshops

2.1 Embeddings Jan 10-14

The workshop was devoted to all aspects of metric embeddings: the general theory of non-linear functional analysis, links between embeddings and geometric group theory and applications of embeddings within theoretical computer science. The workshop featured around 25 lectures with speakers including Noga Alon, Subhash Khot, Nati Linial, Assaf Naor and Laurent Saloff-Coste. Khot and Saloff-Coste gave particularly accessible survey lectures on the Unique Games Conjecture and Random Walks on Groups, respectively. Two of the most intriguing talks for participants were those by Guoliang Yu and Goulnara Arzhantseva on the relationship bewteen coarse embeddings of metric spaces and the Novikov Conjecture for higher signatures.

The questionnaire responses were uniformly positive: all respondents rated the scientific content of the workshop high or above and more than three-quaters rated it excellent. It is always difficult to persuade diverse groups to talk to one another even when they share mathematical techniques and some participants said they would have liked a few more overview lectures, but the general feeling was that the workshop succeeded in making connections between researchers in three different branches of mathematics.

2.2 Discrete Harmonic Analysis Mar 28-Apr 1

The goal of this workshop was to bring together researchers interested in harmonic analysis in a discrete setting: in particular, properties of Fourier-Walsh expansions, influences of variables, Sobolev type inequalities and isoperimetry or concentration estimates. Such analysis plays a role in many fields and has applications to combinatorics, theoretical computer science, probability theory, statistics and number theory. Several talks were devoted to the interplay between the continuous and discrete settings (for example the implementation of log-concavity or curvature methods in the discrete case, geometric properties of Gaussian measures and so on).

The workshop announcement was met with an extremely enthusiastic response and was eventually attended by over 100 participants. There were about 20 lectures with speakers including Gowers, Håstad, Kalai and Wigderson (RVP lecture). Among these the lecture of Wigderson described below must count as the highlight but the most important effect of the workshop was to make it clear to participants how very closely related are the methods being used in the several different fields referred to above.

The questionnaires gave the workshop an 86% excellent rating for scientific content.

2.3 Groups and Additive Combinatorics Jun 27-Jul 1

The final workshop was held at Gregynog Hall, Powys. This proved to be an excellent venue and many favourable comments were received from the participants. The theme of the workshop had been decided in 2010, following a sharp upturn of interest in topics at the interface of group theory and additive combinatorics sparked by a remarkable paper of Hrushovski and subsequent work of Pyber-Szabo and Breuillard-Green-Tao. One principal aim was to introduce group theorists to the new developments, in the hope that they might bring new insights or suggest new directions.

Many of the talks at the workshop concerned approximate groups and their connection with expansion and rapid generation in finite groups. Almost all of the main players in this area were present, gave presentations, and continued their collaborations. Hrushovski gave an admirable account of his model-theoretic approach for this nonspecialist audience. Pyber spoke on his work with Szabó on approximate linear groups, Breuillard spoke on his new (2011) work with Green, Sanders and Tao on approximate groups in general, and Salehi-Golsefidy/Varjú and Guralnick all discussed their ongoing work on expansion phenomena.

Other talks at the workshop amply illustrated the diversity of phenomena that one may hope to study using a combination of group theoretic and additive combinatorial methods. For example, Elon Lindenstrauss discussed his deep work with Bourgain, Mozes and Furman on orbits of $SL_n(R)$ on the torus.

The workshop concluded with an open problems session in which a variety of intriguing questions were presented. The questionnaires gave the workshop an 96% excellent rating for scientific content.

3 Rothschild Lecture

The Rothschild Distinguished Visiting Fellow for the programme was Prof. Avi Wigderson from the IAS in Princeton who is perhaps the world's leading expert on the theory of algorithms. Among his many accolades are the Nevanlinna Prize, the Conant Prize and the Gödel Prize.

Prof. Wigderson delivered the RVP lecture on "The power and weakness of randomness, when you are short on time." The lecture was screened simultaneously in a second room since the main lecture room was unable to accommodate all attendees, including many from the Cambridge Computer Science Department.

The lecture was a brilliantly clear and inspiring account of the way in which randomisation has been used to produce polynomial-time algorithms to solve a host of computational problems for which there is no deterministic algorithm. Prof. Wigderson discussed the

apparently paradoxical fact that if (as everyone believes) the class of NP-hard problems cannot be solved with algorithms that run in a time that is less than exponential as a function of the input size, then it must in principle be possible to remove the randomness: to find algorithms that are actually deterministic. He went on to describe the astonishing applications to Probabilistically Checkable Proofs and Zero Knowledge Proofs. He gave the light-hearted example of a mathematician who wishes to demonstrate to others that s/he possesses a proof of a mathematical theorem without revealing anything about what the proof actually says. The true applications are very much more serious: you wish to demonstrate to retailers that you possess your credit card without revealing anything that would enable the retailer or eavesdroppers to demonstrate that they possess it.

4 Outcomes and Collaborations

At least half the participants described specific collaborations that they had begun or continued at the INI. The following is a selection of some of the collaborative or individual work that has already resulted in articles.

Breuillard, Green, Sanders and Tao proved the nonabelian Freiman conjecture of Helfgott and Lindenstrauss. This is now being written up as a major article in additive combinatorics.

Oleszkiewicz and Sen (a postdoc in Cambridge) found a way to relate their work on hypercontractivity for the discrete cube to the study of queues (a topic which has many applications in communications). This will be published in a joint article with Mossel.

Bennett, Bez and Gutierrez completed two articles on geometric estimates for transforms of Radon type.

Naor completed an article on the Grothendieck constant. This quantity arose in a seemingly very abstract problem in functional analysis but in the last few years has been found to be equal to the gap between classical versus quantum computation for a standard communication model. Naor has shown that the long held belief as to the value of this quantity is false. Another participant, König, has been the expert on this quantity for many years and he and Naor have begun a collaboration on the next steps.

Arzhantseva made use of her stay in Cambridge to work with C. Drutu, who was not a programme participant (but is in the UK), on the Rapid Decay property for word hyperbolic groups.

Braverman, Hatami and Gowers used results on quasi-random groups to improve the best available bounds on a problem in communication complexity. This was a striking example of two different parts of the programme interacting very successfully.

Bulatov, Dyer, Goldberg and Jerrum established a connection between log-supermodular

functions and constraint satisfaction problems in computing.

Solymosi and Tao established a higher-dimensional versions of the famous Szemerédi-Trotter theorem, concerning point-line incidences.