

## On a Relevant Aspect in Difference Equations

By  
Ezio Marchi \*)

Abstract: In this short note, we obtain the solution of the most general linear system of difference. First this was solved in [1], however here we solve it in a much more easy way.

---

\*) Emeritus Professor, Univ. Nac. De San Luis, San Luis, Argentina, Founder and First Director IMASL.  
(ex) Superior Researcher CONICET.  
e-mail: [emarchi1940@gmail.com](mailto:emarchi1940@gmail.com).

Consider the infinite set of nonlinear equation

$$a_{j+1} = \sum_{k=0}^j \alpha_k^{j+1} a_k + \sum_{i=1}^{j-1} \sum_{k=1}^{j-1} \gamma_{i,k}^{j+1} a_i a_k + b_{j+1} \quad \text{for } j > 0$$

Introducing

$${}^1a_j = {}^2a_j + {}^2Q_j$$

it turns out  ${}^1a_j = {}^2q_j$  for  $0 \leq i \leq 2$  because  ${}^1a_{j+1}$  only has nonlinear terms for  $j \geq 2$ .

If we develop  $a_j$  as a function of the initial condition  $a_0$  we have

$$\begin{aligned} {}^2Q_{j+1} &= \left( \sum_{r=1}^{j+1} \sum_{(l_1, l_2, \dots, l_r)}^{j+1} \nabla_{(l_1, \dots, l_r)}^{j+1} \right) q_0 \\ &+ \sum_{i=1}^i \left( \sum_{r=1}^{j+1-2} \sum_{(l_1, \dots, l_r)}^{j+1-2} {}^1\nabla_{(l_1, \dots, l_r)}^{j+1} \right) {}^1b_i \} + {}^1b_{j+1} \end{aligned}$$

Where  $E_r^1$  is the set of all  $r$  – components vectors  $(l_1, \dots, l_r)$  such that

$$\sum_{i=1}^r l_i = l \quad \text{whith} \quad l_i = \text{integer}$$

and

$$\nabla_{(l_1, l_2, \dots, l_r)}^i \cong {}^i\alpha_{j-e_1}^j \quad {}^i\alpha_{j-l_1-l_2}^{j-l_1} \quad {}^i\alpha_{j-l_1-l_2-l_2}^{j-l_1-l_2} \sim {}^i\alpha_{j-l}^{j-\sum_{i=1}^{r-1} l_i}$$

## Introduction

The subject of difference equations is a very important topic from the theoretical point of view and mainly for applications. One good reference is the book by Poole [2].

We remember the reader that if we wish to solve the most general linear problem in difference equations, then we have to solve the following one, namely:

$$x_m = a_{m-1}^m x_{m-1} + a_{m-2}^m x_{m-2} + \dots + a_1^m x_1 + a_e^m x_e + b_m^m$$

This has been solved by induction principle by the authors many years ago. However this approach was difficult.

We give a view of the older method. Now here we are solving it in a easy way.

By the way, we would like to say that many specialists in the subject and some paragraphs in advance and common books, say that this general problem is unsolvable. Moreover, many of these specialists criticize it, since in the computer the memory blows up for some complicated problems.

Our old formula was applied successfully in several subjects in particular to mathematical models in biotechnology.

Now, we have found a new form which is easier than the old one.

In the equation (1) if we revert the order, that is to say

$$x_m = a_0^n x_0 + a_1^m x_1 \dots \dots + a_{m-2}^m x_{m-2} + a_{m-1}^m x_{m-1} + b^m$$

and now take the first  $n$  terms from  $n = 0$  in the form

$$x_1 = a_0^1 x_0 \qquad \qquad \qquad + b^1$$

$$x_2 = a_0^2 x_0 + a_1^2 x_1 \qquad \qquad \qquad + b^2$$

$$x_3 = a_0^3 x_0 + a_1^3 x_1 + a_2^3 x_2 \qquad \qquad \qquad + b^3$$

$$x_{m-2} = a_0^{m-2} x_0 + a_1^{m-2} x_1 + a_2^{m-2} x_2 + \dots \qquad \qquad \qquad + b^{m-2}$$

$$x_{m-1} = a_0^{m-1} x_0 + a_1^{m-1} x_1 + a_2^{m-1} x_2 + \dots + a_{m-2}^{m-1} x_{m-2} \qquad \qquad \qquad + b^{m-1}$$

$$x_m = a_0^m x_0 + a_1^m x_1 + a_2^m x_2 + \dots + a_{m-1}^m x_{m-1} \qquad \qquad \qquad + b^m$$

Now if you take and define the matrix  $m \times m$

$$A_m = \begin{pmatrix} a_0^1 & & & & & & \\ a_0^2 a_1^2 & & & & & & \\ a_0^3 a_1^3 a_2^3 & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ a_0^m a_1^m \dots \dots \dots a_{m-1}^m \end{pmatrix}$$

and the column vector

$$\begin{pmatrix} b^1 \\ b^2 \\ \vdots \\ \vdots \\ \vdots \\ b^m \end{pmatrix}$$

Then the previous system by linear algebra may reduce to

$$\begin{pmatrix} x_1 \\ x_1 \\ \vdots \\ \vdots \\ x_m \end{pmatrix} = A \begin{pmatrix} x_0 \\ x_1 \\ \\ \\ x_{m-1} \end{pmatrix} + b$$

Introducing

$$\bar{x}(r, s) = \begin{pmatrix} x_r \\ \vdots \\ x_s \end{pmatrix}$$

$s > r$ , of length  $s - r$ , the one of the system is transformed to

$$(\bar{x}(1, m) - b) = A(\bar{x}(0, m - 1)) + b$$

From here we have

$$(\bar{x}(1, m) - b) = A\bar{x}(0, m - 1)$$

Now  $A$  is triangular and if the determinant is different of zero, or

$$\prod_{j=1}^m a_{j-1}^j$$

then it is computed and is well defined the inverse of  $A$ . We call  $A^{-1}$  its inverse. Then multiply from the left we have

$$A^{-1}(\bar{x}(1, m) - b) = \bar{x}(0, m - 1)$$

or

$$-A^{-1}b = \bar{x}(0, m - 1) - A^{-1}\bar{x}(1, m)$$

$$-A^{-1}b = \begin{pmatrix} x_0 \\ x_{m-1} \end{pmatrix} - A^{-1} \begin{pmatrix} x_1 \\ x_m \end{pmatrix}$$

$$-A^{-1}b = \begin{pmatrix} x_0 - A_{11}^{-1}x_1 \\ x_1 - A_{22}^{-1}x_2 \end{pmatrix}$$

This given  $x_0$  all the other  $x^1$ s may be easily determined. Therefore this is a solution more general but simple.

## Bibliography

E. Marchi, Leopoldo Millan and Edgar Crinó: Exact Souction of the Rate Equations for the  
threx

Energy Level Laser. TEEE Joy. Of Ausuter Electon Vol QE-IW N.4.

E. Marchi and P. Morillas. The solution of a general polymonial recursive formula.

Ciencia Abierta [http://www.cecuchile.cl/~vof\(2003\)](http://www.cecuchile.cl/~vof(2003)).

Poole D. Algebra lineal (2004) Thompson.

The author would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme 'Discrete Analysis' when work on this paper was undertaken. This work was supported by

**EPSRC Grant Number EP/K032208/1**