Matricial Potentiation

By

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Abstract

In this short note we introduce the potentiation of matrices of the same size. We study some simple properties and some example.

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Introduction

We introduce the matrix potentiation. The problem was pose byGuametti.

Consider two matrices$A$, of size$mxn$and $B,\quad mxn$. We wish to define the potentiation

$$C = A^B$$

For this purpose. We take the logarithm

$$ln \quad C = B \quad ln \quad A,$$

This is valid if $C$and$A$ are not singular. From now on we assome that when we take aln the argument is not singular. The $ln \quad C$ is well defined for the matrices $A$and B real or complex. In the second case we have that it is a multivalued funchon. This assuming that he $ln \quad A$ is a converging sequence. Consider the matrix $D = A - I$ where I stands for the identity matrix. Then

$$ln \quad A = ln \quad (D + I) = D - \frac{D^2}{2} + \frac{D^3}{3} - \frac{D^4}{4} + \frac{D^5}{5} + (-)\frac{^{i+1}D^i}{i}$$

From here, it is inmediated that if $m = n$

$$C = e^{B \quad ln \quad A}$$

can be well defined. The only condition that is necessary the convergence of $ln \quad A$, or

$$ln \quad C = B \quad ln \quad A = B \quad [(A - I) - \frac{(A - I)^2}{2} + \frac{(A - I)^3}{3} \ldots \ldots \ldots \ldots$$

$$= B \quad \left[\sum_{i=1}^{\infty} (-)^{i+1} \quad \left(\frac{A - I}{i}\right)^i\right] = B \quad \left[\sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \quad \sum_{j=0}^{i} \binom{i}{j} \quad A^i (-I)^{i-j}\right]$$

If $A - I$is diagonalizable then

$$A - I = Q^{+1} \quad \Lambda \quad Q^{-1}$$

where the matrix Λ is diagonal with all the eigenvalues in the main diagonal, and Q is formed by the ligenvectors.

Therefore

$$ln \quad C = B \quad ln \quad A = B \quad Q \quad \left[\sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \quad \Lambda^i\right] Q^{-1}$$

or

$$C = e^{B \quad ln \quad A} = e^{B \quad Q\left[\sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \quad \Lambda^i\right] Q^{-1}} = \sum_{t=0}^{\infty} \frac{1}{t!}[B \quad Q \quad K \quad (A) \quad Q^{-1}]^t$$

where

$$K \quad (A) = \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \Lambda^i$$

We have

$$ln \quad A = \sum_{o=1}^{\infty} \frac{(-)^{i+1}}{i} \sum_{j=0}^{i} \binom{i}{j} \quad (-)^{i-j} \quad A^j$$

and it easy to see that A is diagonalizable in the following way

$$A = Q \quad (\Lambda + I) \quad Q^{-1}$$

Therefore

$$\begin{aligned} ln \quad A &= \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \sum_{j=0}^{i} (-)^{i-j} \quad \binom{i}{j} \quad Q \quad (\Lambda + I)^j \quad Q^{-1} \\ &= \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \quad Q \quad \left\{ \sum_{j=0}^{i} (-)^{i-j} \quad \binom{i}{j} \quad (\Lambda + I)^j \right\} \quad Q^{-1} \end{aligned}$$

As an example if A is diagonal: $A = diag(x_1, \quad x_2, \quad \cdots\cdots, \quad x_m)$ them $A^j = diag(x_1, \quad x_2, \quad \cdots\cdots, \quad x_m)$. replacing we see that $ln \quad A$ is also diagonal

$$\begin{aligned} ln \quad A \quad (r,r) &= \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \sum_{j=0}^{i} (-)^i \quad \binom{i}{j} \quad (\lambda_r + 1)^j = \sum_{i=1}^{\infty} (\lambda_r)^i \quad \frac{(-)^{i=1}}{i} \\ &= \sum_{i=1}^{\infty} (\lambda_r + \downarrow -1)^i \quad \frac{(-)^i}{i} = lm \quad \lambda_s \end{aligned}$$

$$ln \quad A \quad (r,s) = r \neq s$$

them$ln \quad A$ is diagonal and it is converging

$$ln \quad \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_m \end{pmatrix} = \begin{pmatrix} ln \quad \lambda_1 & & 0 \\ & ln \quad \lambda_2 & \\ 0 & & ln \quad \lambda_m \end{pmatrix}$$

Next case, we have that A is diagonalizable

$$A = P \quad \Lambda \quad P^{-1}$$

Them the

$$ln \quad A=\sum_{i=1}^{\infty}\frac{(-)^{i+1}}{i}p^{+1}\Lambda^{i}p^{-1}=p^{+1}\sum_{i=1}^{\infty}\frac{(-)^{i=1}}{i}(\Lambda)^{i}p^{-1}=p^{-1}\left[\sum_{i=1}^{\infty}\frac{(-)^{i+1}}{i}\quad(\Lambda)^{i}\right]p^{-1}$$

But$Q=p^{1}$

$$Q\Lambda^{i}=\lambda_1^{i}\begin{bmatrix}q_{11}\\q_{21}\\q_{m1}\end{bmatrix}+\lambda_2^{i}\begin{bmatrix}q_{12}\\q_{22}\\q_{m2}\end{bmatrix}+\cdots=\sum_{r=1}^{n}\lambda_r^{i}\begin{bmatrix}q_{1r}\\q_{2r}\\q_{nr}\end{bmatrix}$$

therefore

$$Q\Lambda \quad P \quad (r,s)=\sum_{k=1}^{n}\lambda_k\, q_{rk}p_{ks}$$

and replacing into equation (1),it twins out that

$$\begin{aligned}(ln \quad A) \quad (r,s)&=\sum_{i=1}^{\infty}\frac{(-)^{i+1}}{i}\quad\left(\sum_{k=1}^{n}(\lambda_k-1)\right)^{i}\quad q_{rk}\quad p_{ks})\\&=\sum_{i=1}^{\infty}\frac{(-)^{i+1}}{i}\quad\left(\sum_{k=1}^{n}(\lambda_k-1)^{i}\right)\quad q_{rk}\quad p_{ks}\\&=\sum_{k=1}^{n}q_{rk}\quad\left[\sum_{i=1}^{\infty}\frac{(-)^{i+1}}{i}\quad(\lambda_k-1)^{i}\right]\quad p_{ks}=\sum_{k=1}^{n}q_{rk}\quad(ln \quad \lambda_k)\quad p_{ks}\\&=[p^{+1}\quad(ln \quad \Lambda)\quad p^{-1}]\quad(r,s)\end{aligned}$$

As a conclusion we have for $A=p^{+1}\quad\Lambda\quad p^{-1}$ diagonalizable with the eigenvalues for Fubini$0<\lambda_r<2$ cocient

$$ln \quad A=p^{+1}\quad(ln \quad \Lambda)\quad p^{-1}$$

and

$$ln \quad \Lambda=\left(\begin{pmatrix}ln \quad \lambda_1 & & c\\ & \ddots & \\ 0 & & ln \quad \lambda_n\end{pmatrix}\right)$$

Then we have proved the following result.

<u>Theorem</u>: Given two diagonalizables matrices mxn: A and B , where $A-I$ has eigenvaluesx_r: $0<x_r$, then the matrix

$$C=A^{B}$$

is well defines and it has the form

$$C = \sum_{t=0}^{\infty} \frac{(B\bar{A})^t}{t!}$$

where

$$\bar{A} = ln \quad A$$

This we have been successful in the definition of matrix potentiation.

<u>Properties</u>

In this section we are going to for diagonalizable matrices some the first one, already proved is

$$A^B \leftrightarrow exp(B \quad ln \quad A)$$

Now we study

$$(A^B)^C$$

We have

$$(A^B)^C = D^C = exp(C \quad ln \quad D) = exp(C \quad ln \quad A^B) = exp(C \quad B \quad ln \quad A) = A^{CB}$$

wher$D = A^B$.

On the other hand

$$A^B A^C = exp(B \quad ln \quad A) exp(C \quad ln \quad A) = exp(B \quad ln \quad A + C \quad ln \quad A)$$
$$= exp((B + C) \quad ln \quad A) = A^{B+C}$$

We follow with

$$exp(ln \quad A) = A$$

We know

$$e^x = \sum_k \frac{x^k}{k^1}$$

an if A_h^k diagonalizable we have

$$ln \quad A = P \quad ln \quad \Lambda \quad p^{-1}$$

where the diagonal matrix Λ is formed by the eigenvalues by the eigevector as columns. Then

$$ln\quad(\exp A) = ln\quad\left(\sum_{k=0}^{\infty}\frac{P\quad\Lambda^k\quad p^{-1}}{k!}\right) = ln\quad\left(P\quad\sum_{k=0}^{\infty}\frac{\Lambda^k}{k!}\quad P^{-1}\right)$$
$$= \sum_{k=1}^{\infty}\frac{(-)^{k-1}}{k}(P^{-1}\quad exp(\Lambda)\quad P)^k = \sum_{k=1}^{\infty}\frac{(-)^{k-1}}{k}\quad P^{-1}\quad (exp\Lambda)^k\quad P$$
$$= P^{-1}\left(\sum_{k=1}^{\infty}\frac{(-)^{k-1}}{k}\quad (exp\Lambda)^k\right)P = P^{-1}\quad\Lambda\quad P = A$$

and in this way we have proved the property.

On the other hand, we have, another basic property, for diagonalizable matrices namely.

Others properties are

$$exp(A+B) = \exp A \exp B$$

Consider

$$e^{(A+B)} = \sum_{k=0}^{\infty}\frac{(A+B)^k}{k!} = \sum_{k=0}^{\infty}\sum_{t=0}^{k}\frac{\binom{k}{t}\quad A^t\quad B^{k-t}}{k!} = \sum_{k=0}^{\infty}\sum_{t=0}^{k}\frac{A^t\quad B^{k-t}}{t!\quad(k-t)!}$$

And now by Fubini property and the standardamegament

$$= \sum_{i=0}^{\infty}\sum_{k=t}^{\infty}\frac{A^t\quad B^{k-t}}{t!\quad(k-t)!}$$

and by a variable change $k-t=j$ and $t=i$ the

$$= \sum_{i=0}^{\infty}\sum_{k=t}^{\infty}\frac{A^i\quad B^j}{i!\quad j!} = \left(\sum_{i=0}^{\infty}\frac{A^i}{i}\right)\quad\left(\sum_{j=0}^{\infty}\frac{B^j}{j!}\right) = \begin{matrix}\exp A\\ \exp B\end{matrix}$$

and is this way the property is proved.

On the other hand, we now prove

$$ln\quad(AB) = ln\quad A + ln\quad B$$

always in the case that $AB = BA$, or they comments let

$$ln\quad(AB) = C$$

$$\exp C = \exp(ln\quad AB) = AB$$

On the other hand if we call

$$D = ln \quad A + ln \quad B$$

then

$$\exp D = \exp(ln \quad A + ln \quad B) = \exp(ln \quad A) . \exp(ln \quad B) = AB$$

then $C = D$.

Now we present

$$A^B A^C = A^{B+C}$$

Let

$$A^B A^C = \exp(B \quad ln \quad A) . \exp(C \quad ln \quad A) = \exp(B \quad ln \quad A + C \quad ln \quad A)$$
$$= \exp((B + C) \quad ln \quad A)) = \exp ln \quad (A)^{B+C} = A^{B+C}$$

Next

$$(A^B)^C = A^{B+C}$$

consider the first term. Calling $D = A^B$then

$$(A^B)^C = D^C = \exp(C \quad ln \quad D) = \exp(C \quad ln \quad A^B) = \exp(C \quad B \quad ln \quad A) = A^{CB}$$

Next we consider a property about the determining namely

$$\det(ln \quad A) = \det(ln \quad \Lambda)$$

When $A = P^{-1} \quad \Lambda \quad P$ which is inmediate by the decomposition.

Consider the transperise of $A : A^t$, then

$$\exp(A^t) = (\exp A)^t$$

Let $A = P^{-1} \quad \Lambda \quad P$ then

$$A^t = (P^{-1} \quad \Lambda \quad P)^t = P^t \quad \Lambda \quad (P^{-1})^t = P^t \quad \Lambda \quad (P^t)^{-1}$$

On the other hand

$$\exp(A^t) = \sum_{i=0}^{\infty} \frac{(A^t)^{-1}}{i!} = P^t \quad \left(\sum_{i=0}^{\infty} \frac{\Lambda^i}{i!} \right) \quad (p^t)^{-1} = \left(P^{-1} \quad \sum_{i=0}^{\infty} \frac{\Lambda^i}{i!} \quad P \right)^t = (e^A)^t$$

Now we consider another property, namely

$$\det \exp A = \exp \operatorname{tra} \Lambda$$

Let

$$e^{A} = P^{-1} \quad \left(\sum_{k=0}^{\infty} \frac{\Lambda^{i}}{i!} \right) \quad P$$

$$\det(e^{A}) = \det\left(\sum_{i=0}^{\infty} \frac{\Lambda^{i}}{i!} \right) = \det(e^{\Lambda}) = \prod_{j=1}^{n} e^{\lambda i} = e^{\lambda_1} e^{\lambda_2} \ldots \ldots \ldots e^{\lambda_n} = e^{\lambda_1 + \lambda_2 \ldots\ldots + \lambda_n} = e^{\operatorname{tra} \Lambda}$$
$$= e^{\operatorname{tra} A}$$

where tra, indicate the trace of the matriz.

Now we consider an example.

$$\begin{pmatrix} \sum_{i=1}^{\infty} \frac{(-)^{i-1}}{i} & \sum_{i=1}^{\infty} \frac{(-)^{i-1}}{i} \quad \left(\frac{1}{2}\right)^{i} \\ 2 \sum_{i=1}^{\infty} \frac{(-)^{i-1}}{i} & \sum_{i=1}^{\infty} \frac{(-)^{i-1}}{i} \quad \left(\frac{1}{2}\right)^{i} \end{pmatrix} = ln \quad C = \begin{pmatrix} 0 & ln \quad {}^{1}/_{2} \\ 0 & ln \quad {}^{1}/_{2} \end{pmatrix}$$

Ej. 2

$$I + \frac{ln \quad C}{1!} + \frac{(ln \quad C)^{2}}{2!} + \frac{(ln \quad C)^{3}}{3!}$$

$$\begin{pmatrix} 0 & ln \quad {}^{1}/_{2} \\ 0 & ln \quad {}^{1}/_{2} \end{pmatrix} \begin{pmatrix} 0 & ln \quad {}^{1}/_{2} \\ 0 & ln \quad {}^{1}/_{2} \end{pmatrix} = \begin{pmatrix} 0 & \left(ln \quad {}^{1}/_{2}\right)^{2} \\ 0 & \left(ln \quad {}^{1}/_{2}\right)^{2} \end{pmatrix}$$

$$(ln \quad C)^{n} = \begin{pmatrix} 0 & \left(ln \quad {}^{1}/_{2}\right)^{n} \\ 0 & \left(ln \quad {}^{1}/_{2}\right)^{n} \end{pmatrix} \begin{pmatrix} 0 & \left(ln \quad {}^{1}/_{2}\right)^{n} \\ 0 & \left(ln \quad {}^{1}/_{2}\right)^{n} \end{pmatrix} \begin{pmatrix} 0 & ln \quad {}^{1}/_{2} \\ 0 & ln \quad {}^{1}/_{2} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \left(ln \quad {}^{1}/_{2}\right)^{nH} \\ 0 & \left(ln \quad {}^{1}/_{2}\right)^{nH} \end{pmatrix}$$

$$C = e^{ln \quad C} = I + \begin{pmatrix} 0 & ln \quad {}^{1}/_{2} \\ 0 & ln \quad {}^{1}/_{2} \end{pmatrix} + \begin{pmatrix} 0 & \left(ln \quad {}^{1}/_{2}\right)^{2} \\ 0 & \left(ln \quad {}^{1}/_{2}\right)^{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \sum_{n=1}^{\infty} \frac{\left(ln \quad ^1/_2\right)^n}{n^1} \\ 0 & \sum_{n=1}^{\infty} \frac{\left(ln \quad ^1/_2\right)^n}{n^1} \end{pmatrix} = \begin{pmatrix} 1 & e^{ln \quad ^1/_2}\overline{-1} \\ 0 & ^1/_2 \end{pmatrix}$$

$$1 + \sum = e^{ln \quad ^1/_2} \qquad = \begin{pmatrix} 1 & -^1/_2 \\ 0 & ^1/_2 \end{pmatrix}$$

<u>Example</u>

We wish to comput

$$C = A^B$$

Consider as example

$$A = \begin{pmatrix} 1 & 2 \\ 0 & ^1/_2 \end{pmatrix} \text{and} B = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$$

knouing

$$ln \quad C = B \quad ln \quad A = B \quad ln \quad \text{P} \quad \Lambda \quad P^{-1} = B \quad P \quad ln \quad \Lambda \quad P^{-1}$$

them

$$ln \quad \Lambda = \begin{pmatrix} ln \quad 1 & 0 \\ 0 & ln \quad ^1/_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -ln \quad 2 \end{pmatrix}$$

and where

$$P = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$$

Therefore

$$ln \quad A = P \quad ln \quad \Lambda \quad P^{-1} = \begin{pmatrix} 0 & ln \quad 16 \\ 0 & -ln \quad 2 \end{pmatrix}$$

hence

$$ln \quad C = B \quad ln \quad A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 0 & ln \quad 16 \\ 0 & -ln \quad 2 \end{pmatrix} = \begin{pmatrix} 0 & ln \quad 16 \\ 0 & ln \quad 64 \end{pmatrix}$$

from here

$$C = e^{B \quad ln \quad A} = \sum_{I=0}^{\infty} \frac{(B \quad ln \quad A)^i}{i!} = \sum_{i}^{\infty} \frac{1}{i!} \begin{pmatrix} 0 & 2^{i+i} \quad 3^{i-1} \quad ln \quad 2 \\ 0 & 3^i \quad ln \quad 2 \end{pmatrix}$$

$$= ln \quad 2 \begin{pmatrix} 0 & \sum_{i}^{\infty} \frac{2 \quad 2^i \quad 3^i \quad 3^{-1}}{i1} \\ 0 & \sum_{i}^{\infty} \frac{3^i}{i1} \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & {}^{2}/_{3} \quad e^6 ln \quad 2 \\ 0 & e^3 \quad ln \quad 2 \end{pmatrix}$$

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