AVERAGE EQUILIBRIUM POINTS

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Abstract

In this brief note, we introduce a general new concept of solution of a general n-person non-cooperative strategy game. This new concept of average equilibrium point generalizes that of equilibrium point. As a particular case the concept of equilibrium point is obtained. We prove a general theorem of existence for any average of an average equilibrium point in any mixed extension of a finite n-person game. Moreover some further cases are also studied

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1 Introduction

Consider an n-person finite non-cooperative game given in standard form:

$$\Gamma = \{\Sigma_i; A_i; i \in N\}$$

where $N = \{1, 2, 3, \ldots, n\}$ is the set of players. Σ_i denotes the set of strategies of the players $i \in N$, wich is considered to be non-empty and finite. Finally

$$A_i : \bigtimes_{j=1}^{n} \Sigma_j \to R$$

denotes the payoff of the players $i \in N$. Here R stand for set of real numbers.

The mixed extension of σ indicates by

$$\widetilde{\Gamma} = \{\widetilde{\Sigma}_i : E_i; i \in N\}$$

is formed by the sets $\widetilde{\Sigma}_i$ of mixed strategies over E_i, for any $i \in N$, and the expectation E_i defined on $\bigtimes_{j \in N} \widetilde{\Sigma}_j$

This settings of finite n-person game and its mixed extension $\widetilde{\Sigma}$, can be extended for more generalized situations, namely we introduce the concept of general game.

$$\Gamma = \{\Sigma_i; A_i; i \in N\}$$

where the strategy set σ_i is a non-empty closed convex and bounded in an euclidean space.

On the other hand, the payoff function A_i is considered by simplicity continuous. Then we remind that a real continuous function

$$f : \Sigma \subset R^n \to R$$

defined as a non-empty, convex, compact set σ in a euclidean space, is said to be quasi-concave if the sets

$$f_\lambda = \{\sigma \in \Sigma : f(\sigma) \geq \lambda)\}$$

for any real λ are convex.

So in this way, we present the Nash's important result:

Given any general n-person game where the payoff functions of each player is quasi-concave in the variable

$$\sigma_i \in \Sigma_i$$

for any $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n) \in \bigtimes_{j \neq i} \Sigma_j$, there exists an equilibrium point, namely:

$$\bar{\sigma} = (\bar{\sigma}_i, \bar{\sigma}_{-i}) : A_i(\sigma) \geq A_i(\sigma_i, \bar{\sigma}_{-i}) \qquad \forall_i \forall_{\sigma_i}$$

This important concept has been generalized in several directions. For example, recently, we have defined the friendly equilibrium points and we have derived generalization of this result in normal and extensive games. Selten [6] was the first of refined the perfect equilibrium point to proper and then Myerson [4] refined to perfect equilibrium. Further there have been further refinement as for example that of Garcia Jurado [2]

2 Average Equilibrium Points

Now in this section, we extend the concept of equilibrium point to that of average equilibrium point.

We have some different kinds of new concepts. First we define the strict average equilibrium of a general non-cooperative n-person game

$$\sigma = \{\Sigma_i; A_i; i \in N\}$$

where Σ_i is a non-empty, compact and convex subset in an euclidean space. The payoff functions A_i are linear in the variable σ_i for $\sigma_{-i} \bigtimes_{j \neq i} \Sigma_j$:

$$A_i(\cdot\,, \sigma_{-i}) \qquad \text{linear}$$

Therefore consider the average weight α_i, β_i with the property $0 \leq \alpha_i, \beta_i \leq 1$ and $\alpha_i + \beta_i = 1$

A point $\bar{\sigma} = (\bar{\sigma}_1, \ldots, \bar{\sigma}_n) \in \bigtimes_{j=1} \Sigma_j$ is called (α_i, β_i)-average equilibrium point of the game σ if

$$\alpha_i \max_{\rho_i} A_i(\rho_i, \bar{\sigma}_{-i}) + \beta_i \min_{\rho_i}(\rho_i, \bar{\sigma}_{-i}) = A_i(\bar{\sigma}_i, \bar{\sigma}_{-i}) \qquad \forall_i \forall_{\sigma_i}$$

Theorem 1 *Given σ, where $A_i(\cdot\,, \bar{\sigma}_{-i})$ is linear in σ_i for each σ_{-i}, and a set of average (α_i, β_i) for each player, then there always exists an (α_i, β_i) average equilibrium point*

Proof. Consider for an arbitrary $\sigma = \bigtimes_{j \in N} \Sigma_j$ the set:

$$\psi_i(\sigma) = \{\tau_i \in \Sigma_i : A_i(\tau_i, \sigma_{-i}) = \alpha_i \max_{\rho_i} A_i(\rho_i, \sigma) + \beta_i \min_{\rho_i} A_i(\rho_i, \sigma_{-i})\}$$

This set is non-empty, convex and compact, since A_i is continuous and linear in $\bar{\sigma}_i$. Take $\bar{\rho}_i$ as a point reaching the maximum and ρ_i reaching the minimum. Therefore a point defined as $\tau_i = \alpha_i \bar{\rho}_i + \beta_i \rho_i$ belongs to $\psi_i(\sigma)$. Taking the Cartesian product

$$\psi(\sigma) = \bigtimes_{i=1}^{n} \psi_i(\sigma)$$

we define a multivalent application from $\bigtimes_{i=1}^{n} \Sigma_i$ to the same set. This application is non-empty, compact and convex. Now if we prove that its graph is closed, then we can apply Kakutani's fixed point theorem. This theorem assures that under the mentioned condition there exists a fixed point $\bar{\sigma}$, fulfilling

$$\bar{\sigma} \in \psi(\bar{\sigma})$$

such a point is indeed an (α_i, β_i) average equilibrium point of the game σ.

We need to prove that the graph of ψ is closed. Consider a converging sequence of points $\sigma(n) \in \underset{i \in N}{\times} \Sigma_i$

$$\sigma(n) \to \sigma = (\sigma_i, \sigma_i)$$

then for any given player $i \in N$ and any converging sequence of strategies $\tau_i(n) \in \psi((\sigma(n))$, we have from the definition of $\psi((\sigma(n))$ that

$$A_i(\tau_i(n), \sigma_{-i}(n) = \alpha_i \max_{\rho_i} A(\rho_i, \sigma_{-i}(n)) + \beta \min_{\rho_i} A_i(\rho_i, \sigma_{-i}(n))$$

Then we need to prove that a limit point $\tau_i : \tau_i(n) \to \tau_i$ together of σ_{-i} belongs to $G\psi_i$ or equivalently

$$A_i(\tau_i, \tau_{-i}) = \alpha_i \max_{\rho_i} A_i(\rho_i, \sigma_{-i}) + \beta_i \min_{\rho_i} A_i(\rho_i, \sigma_{-i})$$

But for both given converging sequences

$$\sigma(n) \to \sigma \qquad \text{and} \qquad \tau(n) \to \tau$$

by the continuity of the payoff functions A_i we have that

$$\lim_{n\to\infty} \max_{\rho_i} A_i(\rho_i, \sigma_{-i}(n)) = \max_{\rho_i} A_i(\rho_i, \sigma_{-i})$$

$$\lim_{n\to\infty} \min_{\rho_i} A_i(\rho_i, \sigma_{-i}(n)) = \min_{\rho_i} A_i(\rho_i, \sigma_{-i})$$

and

$$\lim_{n\to\infty} A_i(\tau_i, \sigma_{-i}) = A_i(\tau_i, \sigma_{-i})$$

moreover, from here we obtain

$$A_i(\tau_i, \sigma_{-i}) = \alpha_i \max_{\rho_i} A_i(\rho_i, \sigma_{-i}) + \beta_i \min_{\rho_i} A_i(\rho_i, \sigma_{-i})$$

for each payoff function.

Therefore we have proved that the graph ψ is closed. In this way we have satisfied all requirements of Kakutani's fixed point theorem. Thus the existence of a fixed point $\bar{\sigma}$ for the multivalued function ψ is guaranteed: $\bar{\sigma} \in \psi(\bar{\sigma})$. But this in term of the payoff functions says

$$A_i(\bar{\sigma}_i, \bar{\sigma}_{-i}) = \max_{\rho_i} A_i(\rho_i, \sigma_{-i}) + \beta_i \min_{\rho_i} A_i(\rho_i, \sigma_{-i})$$

for each player $i \in N$. This is an average equilibrium point for the game σ.(q.e.d).

Before we provide an intuitive idea of solution together with the stability for the strategic situation described by the game σ, we remark that mathematically it is possible to extend the concept of average equilibrium point, by allowing the averages for each player (α_i, β_i) to be function of the strategy $\sigma_i : \big(\alpha_i(\sigma_{-i}), \beta_i(\sigma_{-i})\big)$. In order to have the existence of a functional average point in the game it is needed that $\alpha_i(\sigma_{-i}(n)) \to \alpha_i(\sigma_{-i})$ and $\beta_i(\sigma_{-i}(n)) \to \beta_i(\sigma_{-i})$. From an intuitive point of view, here we have a fact the average change for a given player — the approach of the strategies of all the remaining players.

Now we will provide an intuitive argument for the concept of averaging equilibrium strategy. Remember that the intuitive argument of the concept of equilibrium point for a non-cooperative game, is that, for a given player $i \in N$, if all the remaining players coincide to play σ_{-i}, then player $i \in N$ it is better off if choses σ_i such that he obtain a maximum:

$$A_i(\sigma_i, \sigma_{-i}) = \max_{\rho_i} A_i(\rho_i, \sigma_{-i})$$

This intuitive motivation is essential for taking the equilibrium point concept as fundamental in game theory.

However several authors explain as a second important aspect, the stability fact. This is explained as follows: Assuming that all the decide to play $\sigma = (\sigma_1, \ldots, \sigma_n)$ and this point is an equilibrium point. Therefore (it seems apparently) since player $i \in N$ reaches it maximum, he does not possess any argument to change from it: σ_i. The reason of it is that if he changes, he might obtain less payoff.

However, it is important to emphasis that the previous argument might lack an important aspect of stability. Consider that for the choice of player $i \in N$ assuming known σ_{-i}, the player has to consider the two more important aspects, namely the optimization and the stability. The concept of equilibrium point takes in consideration only the first one. However, the second aspect to be considered of stability, perhaps it is the most important. Assuming for any reason a player $j \neq i$ changes his strategy σ_j to lower one, in such a situation that the might be no rational in the sense that he might not attach his maximum.

$$A_j(\tau_j, \sigma_i, \sigma_{N-\{i,j\}}) = \max_{\rho_j} A_i(\rho_j, \sigma_i, \sigma_{N-\{i,j\}})$$

where $\sigma_{N-\{i,j\}}$ indicates

$$\sigma\big(\sigma_j, \sigma_i, \sigma_1, \ldots, \sigma_{j-i}, \sigma_{j+1}, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n\big)$$

but we the effect that for the player $i \in N$ the payoff at the new point is destructive, that is to say

$$A_i(\sigma_i, \tau_j, \sigma_{N-\{i,j\}}) \ll A_i(\sigma_i, \sigma_{-i}) = \max_{\rho_i} A_i(\rho_i, \sigma_{-i})$$

Thus, these facts show the global importance of cautions to give weights to the fundamental concepts of optimization and stability.

For the reader information, we would like to say that already in the literature it was introduced by us Marchi E. [3] the concept of stable equilibrium, however also this concept has some objections regarding the fact that the stability is considered in a strong way because it is very good but we loose of the payoff. On the other hand the concept of equilibrium point is very good for the payoff but it might be very acquard for the sense of stability. Thus, we are faced with the dual problem how we tackle this multicriteria optimization. The interested reader will referred to the book by Steuer [7]. From here one can get many different concepts of solutions which generally the most important aspect is the Pareto optimality. We sketch new issues.

Indeed if we have $\alpha_i(\sigma_i)$ and $\beta_i(\sigma_{-i})$, then one takes as final result an optimal solution a multicriteria optimization of optimal point described in Stener[].

On the other hand if the α_i and β_i are functions of σ_i then we are faced with the fact that

$$\alpha_i(\sigma_i)A_i(\sigma_i, \sigma_{-i})$$

as well as

$$\beta_i(\sigma_i)A_i(\sigma_i, \sigma_{-i})$$

are product of function. Therefore we suggest that the reader consider the simple contribution by Marchi E. [3] and Flouras [1] for possible extensions.

An example

Take an example of two by two elements having the structure for the payoff as following:

$$A_1 : \begin{array}{|cc|} a_{11} & 0 \\ 0 & a_{22} \end{array} \qquad A_2 : \begin{array}{|cc|} b_{11} & 0 \\ 0 & b_{22} \end{array}$$

Therefore the expectation function are of the form:

$$\begin{aligned} E_1(x,y) &= a_{11}xy + a_{22}(1-x)(1-y) = a_{11}xy + a_{22} - xa_{22} - ya_{22} + a_{22}xy \\ &= x\left[(a_{11}, a_{22}y - a_{22})\right] - a_{22}y + a_{22} \qquad\qquad x \in \tilde{\Sigma}_1, \quad y \in \tilde{\Sigma}_2 \\ E_2(x,y) &= b_{11}xy + b_{22}(1-x)(1-y) = b_{11}xy + b_{22} - xb_{22} - yb_{22} + b_{22}xy \\ &= y\left[(b_{11}, b_{22}x - b_{22})\right] - b_{22}x + b_{22} \end{aligned}$$

If we draw the set $\tilde{\Sigma}_1 \mathbf{x} \tilde{\Sigma}_2$ as follows:

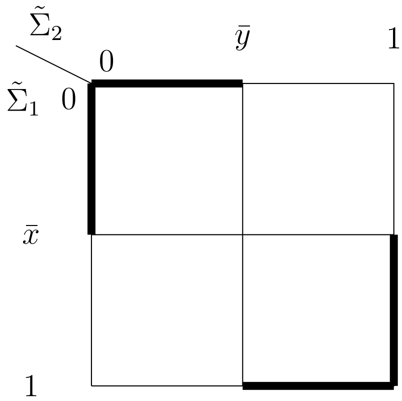

Fig. 1

then where

$$(a_{11} + a_{22})\, y - a_{22} = 0$$

or

$$\bar{y} = \frac{a_{22}}{a_{11} + a_{22}}$$

In the case that $a_{22} > 0$ and $a_{11} > 0$ then $0 < \bar{y} < 1$, and for that point, for all x the payoff function E_1 is maximum since it is constant. For

$$y < \frac{a_{22}}{a_{11} + a_{22}}$$

the payoff function E_1 reaches a maximum at the point $x = 0$ since the first number is negative, and the remaining do not depend on them. Finally, when

$$y > \frac{a_{22}}{a_{11} + a_{22}}$$

E_1 reaches the maximum at $x = 1$

Now we consider the second payoff function. Consider the member

$$(b_{11} + b_{22})\, x - b_{22} = 0$$

then

$$x = \frac{b_{22}}{b_{11} + b_{22}}$$

In this case $b_{22} > 0$ and $b_{11} > 0$ the $0 < \bar{x} < 1$ and for that case, for all the y the payoff function E_2 is maximum since it is constant. For

$$x < \frac{b_{22}}{b_{11} + b_{22}}$$

the payoff function E_2 reaches a maximum at the point $y = 0$ since the first number is negative, and the remaining do not depend on them. Then the last case, when:

$$x > \frac{b_{22}}{b_{11} + b_{22}}$$

E_2 reaches the maximum at $y = 1$.

Consequently, there are three equilibrium points, namely $x = y = 0$, $x = y = 1$ and an interior one

$$\left(\frac{b_{22}}{b_{11} + b_{22}}, \frac{a_{22}}{a_{11} + a_{22}} \right)$$

In this way we found is rather elementary way all the equilibrium points of our game. The powerful and excellent method, we have followed is due to Winkels (1979). Perhaps in the future by extending in a suitable way this important method it would possible to have important new approaches.

Now we will verify that the points suggested by the previous method are indeed equilibrium points. The first one

$$\begin{aligned} E_1(0,0) &= a_{22} \geq E_1(x,0) = -xa_{22} + a_{22} = a_{22}(1-x) && 0 \leq x \leq 1 \\ E_2(0,0) &= b_{22} \geq E_2(0,y) = b_{22}(1-y) && 0 \leq y \leq 1 \end{aligned}$$

and indeed it is an equilibrium point.

On the other hand, now we try with

$$\begin{aligned} E_1(1,1) &= a_{11} \geq E_1(x,1) = a_{11}x && 0 \leq x \leq 1 \\ E_2(1,1) &= b_{11} \geq E_2(1,y) = b_{11}y && 0 \leq y \leq 1 \end{aligned}$$

and again it is an equilibrium point. Finally we try to verify with the interior one:

$$\left(\frac{b_{22}}{b_{11}+b_{22}}, \frac{a_{22}}{a_{11}+a_{22}}\right)$$

$$\begin{aligned} E_1\left(\frac{b_{22}}{b_{11}+b_{22}}, \frac{a_{22}}{a_{11}+a_{22}}\right) &= -a_{22}\frac{a_{22}}{a_{11}+a_{22}} + a_{22} \\ &= a_{22}\left(\frac{-a_{22}}{a_{11}+a_{22}} + 1\right) = a_{22}\frac{a_{11}}{a_{11}+a_{22}} \\ &\geq E_1\left(x, \frac{a_{22}}{a_{11}+a_{22}}\right) = -a_{22}\frac{a_{22}}{a_{11}+a_{22}} + a_{22} \\ &= a_{22}\left(\frac{-a_{22}}{a_{11}+a_{22}} + a_{22}\right) = a_{22}\frac{a_{11}}{a_{11}+a_{22}} \end{aligned}$$

On the other hand, finally

$$\begin{aligned} E_2\left(\frac{b_{22}}{b_{11}+b_{22}}, \frac{a_{22}}{a_{11}+a_{22}}\right) &= -b_{22}\frac{b_{22}}{b_{11}+b_{22}} + b_{22} \\ &= b_{22}\left(\frac{-b_{22}}{b_{11}+b_{22}} + 1\right) = b_{22}\frac{b_{11}}{b_{11}+b_{22}} \\ &\geq E_2\left(\frac{b_{22}}{b_{11}+b_{22}}, y\right) = -b_{22}\frac{b_{22}}{b_{11}+b_{22}} + b_{22} \\ &= b_{22}\left(\frac{-b_{22}}{b_{11}+b_{22}} + b_{22}\right) = b_{22}\frac{b_{11}}{b_{11}+b_{22}} \end{aligned}$$

and so we have proved and verified all the equilibrium points.

Now we can try to see what happen and to study the average equilibrium point.

We just take the same game. The max is provided in the Fig. 1

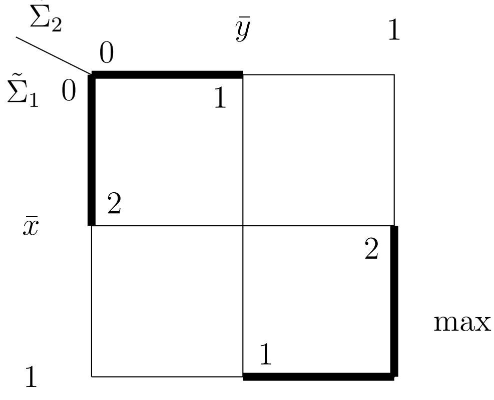

Fig. 2

It would be possible to write down the geometric features, but for argument of completeness and to concern the reader, we are going to go over for the minimum.

We recall that

$$E_1(x, y) = x\left[(a_{11} + a_{22})y - a_{22}\right] - a_{22}y + a_{22}$$

and

$$E_2(x, y) = y\left[(b_{11} + b_{22})x - b_{22}\right] - b_{22}x + b_{22}$$

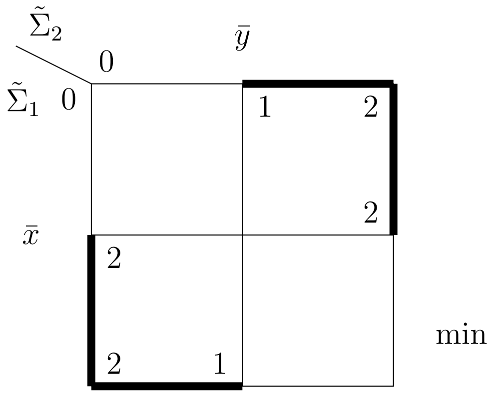

Fig. 3

At the point

$$\bar{y} = \frac{a_{22}}{a_{11} + a_{22}}$$

If $a_{22} > 0$ then $0 < \bar{y} < 1$ and for that point, for all x the payoff E_1 is minimum since it is constant. For

$$y < \frac{a_{22}}{a_{11} + a_{22}}$$

the payoff function E_1 reaches the minimum at the point $x = 1$. Finally if

$$y > \frac{a_{22}}{a_{11} + a_{22}}$$

then the payoff function E_1 reaches the minimum at the point $x = 0$.

Now consider the case of the payoff function for the second player. At the point

$$\bar{x} = \frac{b_{22}}{b_{11} + b_{22}}$$

If $b_{22} > 0$ and $b_{11} > 0$ then $0 < \bar{x} < 1$ for all y the payoff E_2 is reaches a minimum since it is constant. For

$$x < \frac{b_{22}}{b_{11} + b_{22}}$$

the payoff function E_2 reaches the minimum at the point $y = 1$. The last point to considered is when

$$x > \frac{b_{22}}{b_{11} + b_{22}}$$

then the payoff E_2 reaches the minimum at the point $y = 0$.

Now we will present and complete explicatively an average equilibrium point. In general it is very complicate the explicit computation of an arbitrary one. We do not want in this paper to attack the general presentation, however we study a rather simple one.

Consider in the graphs presented in fig 2 and 3 and take

$$E_1(0,0) = a_{22}$$
$$E_2(1,0) = -a_{22} + a_{22} = 0$$

then the average equilibrium point for the component of the function player it is:

$$\alpha_1 a_{22} = E_1(x,0) = -a_{22} + a_2 2$$

then

$$\alpha_1 = (1-x)$$

On the other hand

$$E_2(x,0) = \max \lim_{2} E_2(2,0)$$

provided that

$$(1-x) \leq \bar{x} = \frac{b_{22}}{b_{11} + b_{22}}$$

Thus, the point $(x,0)$ is an average equilibrium point with weights

$$\alpha_i = (1-x), \quad \beta_i = x, \quad \alpha_2 = 1, \quad \beta_2 = 0$$

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