Singularizing Successor Cardinals by Forcing *†‡

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April 26, 2016

Abstract

We exhibit models of set theory, using large cardinals and forcing, in which successor cardinals can be made singular by some "Namba-like" further set forcing, in most cases without collapsing cardinals below that successor cardinal. For successors of *regular* cardinals, we work from consistency-wise optimal assumptions in the ground model. Successors of *singular* cardinals require stronger hypotheses. Our partial orderings are different from Woodin's stationary tower forcing, which requires much stronger hypotheses when singularizing successors of regular cardinals, and collapses cardinals above the cardinal whose cofinality is changed when singularizing successors of singular cardinals.

1 Introduction and statements of results

In Zermelo Fraenkel set theory (ZFC), cardinals κ are either *regular* ($\mathrm{cof}(\kappa) = \kappa$) or *singular* ($\mathrm{cof}(\kappa) < \kappa$), where cof is Hausdorff's cofinality function (see [10]). There are pronounced combinatorial differences between these two classes of cardinals. For example, the value of the cardinal power 2^κ at singular κ is markedly influenced by cardinal exponentiation below κ, whereas for regular κ, 2^κ can be made arbitrarily large by the forcing method, rather independent of behavior below κ. There are a number of classical forcing constructions in which a particular combinatorial situation is first prepared at a regular cardinal which is then made singular by some singularization forcing, thus transferring some possibilities of regular cardinal combinatorics to singular cardinals (see [10] for further details). These constructions typically use regular *limit* cardinals κ, i.e., inaccessible cardinals, for which a variety of singularization forcings have been defined, which preserve κ and other cardinals as cardinals when stepping into the generic extension.

By contrast, singularizing a *successor* cardinal κ^+ by forcing will destroy κ^+ as a cardinal, since successor cardinals are always regular in ZFC. But it is a challenge to singularize to some small value of cofinality and preserve the cardinal κ and possibly smaller ones in the process. Namba forcing [15, 10], also introduced by Bukovský [4], singularizes the successor cardinal $\aleph_2 = \aleph_1^+$ without collapsing $\aleph_1$; for a ground model M, it yields a generic extension $N \supseteq M$ such that $\aleph_1^M = \aleph_1^N$ and $\mathrm{cof}^N(\aleph_2^M) = \aleph_0$. Such results provide important limitations for covering properties in the spirit of Jensen [6] between the set theoretical universe and its constructible inner models.

We describe "Namba-style singularizability" by forcing over M in

Definition 1 $\mathrm{Sing}(\mu,\delta)$ *stands for:* $\delta < \mu$ *are cardinals, and there is a set forcing* $\mathbb{P}$ *such that the following properties are forced by* $\mathbb{P}$*:*

a) Cardinals $<\mu$ *and their cofinalities are preserved between* V *and* $V[\dot{G}]$*, where* $\dot{G}$ *is a canonical name for a generic filter for* $\mathbb{P}$*;*

*2010 Mathematics Subject Classifications: 03E35, 03E45, 03E55.

†Keywords: Namba forcing, singularization, Prikry forcing, measurable cardinal, Woodin cardinal, strongly compact cardinal, supercompact cardinal, consistency strength, core model.

‡This paper was written in part while the authors were Visiting Fellows at the Isaac Newton Institute for Mathematical Sciences in the programme "Mathematical, Foundational and Computational Aspects of the Higher Infinite (HIF)", held from 19 August 2015 until 18 December 2015 and funded by EPSRC grant EP/K032208/1. The second author's participation would not have been possible without the generous support of Dean Jeffrey Peck of Baruch College's Weissman School of Arts and Sciences and Professor Warren Gordon, Chair of the Baruch College Mathematics Department, both of whom the author thanks.

§The second author's research was partially supported by PSC-CUNY grants.

b) $\mathrm{cof}^{V[\dot{G}]}(\mu) = \delta$.

In ZFC, Namba forcing witnesses $\mathrm{Sing}(\aleph_2, \aleph_0)$. If μ is a measurable cardinal, Prikry forcing witnesses $\mathrm{Sing}(\mu, \aleph_0)$. By combining Lévy collapse forcing and Prikry forcing, we can force the singularizability of successor cardinals above $\aleph_2$.

Theorem 2 *Consider a ground model V in which μ is a measurable cardinal and κ is a regular cardinal such that $\aleph_1 < \kappa < \mu$. Then there is a two-stage forcing extension $V \subseteq M \subseteq N$ such that*

a) *V, M, and N possess the same bounded subsets of κ;*

b) *cardinals $\leq \kappa$ are absolute between V and N, and κ is regular in N;*

c) *$\mu = (\kappa^+)^M$;*

d) $\mathrm{cof}^N(\mu) = \aleph_0$.

Hence M satisfies $\mathrm{Sing}(\kappa^+, \aleph_0)$.

Using higher cofinality singularization forcings of Gitik [7], Theorem 2 can be generalized to uncountable cofinalities.

Theorem 3 *Consider a ground model V of GCH with $\aleph_1 \leq \delta < \mu$, where δ is a regular cardinal and μ is a measurable cardinal of Mitchell order $o(\mu) = \delta$. Then there is a two-stage forcing extension $V \subseteq M \subseteq N$ such that δ remains regular in M, μ remains measurable in M, and for κ a fixed regular cardinal in M, $\delta < \kappa < \mu$, in N*

a) *κ and δ are regular;*

b) *$\mu = \kappa^+$;*

c) $\mathrm{Sing}(\kappa^+, \delta)$ *holds.*

Theorems 2 and 3 work from optimal large cardinal assumptions by the following lower bounds on consistency strengths.

Theorem 4 *Assume* $\mathrm{Sing}(\kappa^+, \delta)$*, where $\delta < \kappa$ are regular cardinals and $\kappa \geq \aleph_2$. Then*

a) *if $\delta = \aleph_0$, κ^+ is a measurable cardinal in some inner model;*

b) *if $\delta > \aleph_0$, κ^+ is a measurable cardinal of Mitchell order δ in some inner model.*

For singular cardinals κ, our forcings singularizing κ^+ are achieved from considerably stronger large cardinals.

Theorem 5 *Let $\kappa = \sup_{i<\omega} \kappa_i$, where $\langle \kappa_i \mid i < \omega \rangle$ is a strictly increasing sequence of κ^+-strongly compact cardinals. Then* $\mathrm{Sing}(\kappa^+, \aleph_0)$ *holds via a κ^{++}-c.c. partial ordering which adds no bounded subsets of κ.*

Theorem 6 *Let κ be κ^+-strongly compact. Then there is a forcing extension in which κ is a singular cardinal satisfying* $\mathrm{Sing}(\kappa^+, \aleph_0)$ *via a κ^{++}-c.c. partial ordering which adds no bounded subsets of κ.*

Using Magidor's work from his paper [13], it is possible to transfer the results of Theorem 6 down to $\aleph_\omega$ and $\aleph_{\omega_1}$.

Theorem 7 *Let κ be κ^+-supercompact. Then there is a forcing extension satisfying* $\mathrm{Sing}(\aleph_{\omega+1}, \aleph_0)$ *via a partial ordering which adds no bounded subsets of $\aleph_\omega$ and preserves cardinals $\geq \aleph_{\omega+2}$.*

Theorem 8 *Let κ be κ^{++}-supercompact in a ground model satisfying $2^{\kappa^+} = \kappa^{++}$. Then there is a forcing extension with a partial ordering $\mathbb{P}$ such that forcing with $\mathbb{P}$ changes the cofinality of $\aleph_{\omega_1+1}$ to $\aleph_1$ while preserving $\aleph_{\omega_1}$ and all cardinals $\geq \aleph_{\omega_1+2}$.*

The necessity of strong assumptions for Theorems 5 – 8 follows from

Theorem 9 *Suppose that it is possible to define a set partial ordering $\mathbb{P}$ such that for some singular cardinal κ*

a) κ *remains a singular cardinal in $V^{\mathbb{P}}$;*

b) $(\kappa^+)^V < (\kappa^+)^{V^{\mathbb{P}}}$.

Then there must be an inner model with a Woodin cardinal.

Note that singularizations of successor cardinals are also possible with Woodin's stationary tower forcing (see [12]). Assuming the existence of a Woodin cardinal λ in a model V of ZFC, the stationary tower forcing $\mathbb{P}_{<\lambda}$ can be defined so that for any regular cardinals $\kappa_1 < \kappa_2 < \lambda$, in $V^{\mathbb{P}_{<\lambda}}$ we have that

a) cardinals below κ_2 are preserved;

b) $\mathrm{cof}(\kappa_2) = \kappa_1$;

c) if $\delta < \kappa_2$ is such that $2^\delta < \kappa_2$ in V, then $V^{\mathbb{P}_{<\lambda}}$ contains the same subsets of δ as V.

However the chain condition on $\mathbb{P}_{<\lambda}$ is large, and forcing with $\mathbb{P}_{<\lambda}$ collapses many cardinals above the cardinal being singularized. This is in sharp contrast to Theorems 5 – 8, where no cardinal above the one being singularized is collapsed. In addition, the hypotheses of the existence of a Woodin cardinal λ used in the definition of $\mathbb{P}_{<\lambda}$ are far beyond what is needed to singularize the successor of a regular cardinal, as Theorems 2 – 4 show.

This paper was initiated by the PhD-project of the first author [1], where Theorems 2 and 3 were proved by introducing some Prikry or Magidor-type tree forcings with built-in Lévy collapses and analyzing generic extensions as two-stage iterations. In Section 3 of this paper, we replace the tedious combinatorial work with tree forcings by a general lemma on a "weak commutativity" of Lévy collapses and some singularization forcings, and obtain Theorems 2 and 3 as corollaries. We prove Theorems 4 and 9 in Section 2 as immediate consequences of certain covering theorems. Successors of singular cardinals are treated in Section 4. We conclude with remarks and further questions in Section 5.

2 Lower bounds

We turn our attention now to obtaining lower bounds in consistency strength for the hypotheses needed to define our singularization partial orderings.

For the proof of Theorem 4a), suppose $\aleph_2 \leq \kappa$. Let $\mu = (\kappa^+)^V$. Assume $\mathrm{Sing}(\mu, \aleph_0)$ holds in V and is witnessed via $\mathbb{P}$. We work below $0^{\P}$ (the sharp for a strong cardinal), and take $K =_{\mathrm{df}} K^V$ as the core model below $0^{\P}$. (If $0^{\P}$ exists, then all uncountable cardinals of V, in particular μ, are measurable in some inner model.) Let G be V-generic for $\mathbb{P}$. Since $\mathbb{P}$ is set forcing, $K = K^V = K^{V[G]}$, and $V[G]$ does not contain $0^{\P}$. Further, since $\mathrm{Sing}(\mu, \aleph_0)$ implies that forcing with $\mathbb{P}$ changes μ's cofinality to ω but preserves all cardinals and cofinalities $\leq \kappa$, in $V[G]$, $\mathrm{cof}(\mu) = \omega$, $|\mu| = \kappa$, and $\aleph_2 \leq \kappa < \mu$. In addition, since μ is regular in V and $K \subseteq V$, μ is regular in K. Putting all of this together, we thus have that in $V[G]$, $\aleph_2 < \mu$, μ is regular in K, and $\mathrm{cof}(\mu) < |\mu|$. By [5, Theorem 1], μ is therefore measurable in K. This completes the proof of Theorem 4a).

$\square$

Theorem 4b) follows from (and in fact, is explicitly stated as part of) [14, Theorem 0.1].

$\square$

For the proof of Theorem 9, let $\mathbb{P} \in V$ and κ be as in the hypotheses for Theorem 9. Assume that there is no inner model with a Woodin cardinal. Then by the work of [11], it is possible to build K within V and assume that it satisfies standard facts about core models. In particular, we know that K computes successors of singular cardinals correctly. This means that

$$(\kappa^+)^K = (\kappa^+)^V.$$

Consider now $V^{\mathbb{P}}$. By the absoluteness of K and its properties under set forcing, it is possible to build the same K within $V^{\mathbb{P}}$. Since by assumption (a), κ remains a singular cardinal in $V^{\mathbb{P}}$, it is still the case that

$$(\kappa^+)^K = (\kappa^+)^{V^{\mathbb{P}}}.$$

However, since by assumption (b),

$$(\kappa^+)^V < (\kappa^+)^{V^{\mathbb{P}}},$$

we have that

$$(\kappa^+)^K = (\kappa^+)^V < (\kappa^+)^{V^{\mathbb{P}}} = (\kappa^+)^K.$$

This contradiction completes the proof of Theorem 9.

$\square$

3 Singularizing successors of regular cardinals

The results in this section are based on a lemma about the Lévy collapse

$$\mathrm{Coll}(\kappa, {<}\mu) = \{f \mid f : \kappa \times \mu \to \mu, \forall \langle \xi, \nu \rangle \in \mathrm{dom}(f)[f(\langle \xi, \nu \rangle) < \nu], |\mathrm{dom}(f)| < \kappa\},$$

partially ordered by $p \leq q$ iff $p \supseteq q$. The term $\mathrm{Coll}(\kappa, {<}\mu)$ can be interpreted in various models of ZFC.

Fix a transitive ground model M of ZFC and let μ be inaccessible in M. Let $\kappa < \mu$ be regular in M.

Lemma 10 *Let $\mathbb{P} \in M$ be a partial ordering which does not add bounded subsets of μ. Let G be M-generic for $\mathbb{P}$. Let H be $M[G]$-generic for $(\mathrm{Coll}(\kappa, <\mu))^{M[G]}$. Then*

$$\bar{H} = H \cap (\mathrm{Coll}(\kappa, <\mu))^M$$

is an M-generic filter for $(\mathrm{Coll}(\kappa, <\mu))^M$. Hence, the two-stage forcing iteration $M[G][H]$ is also of the form

$$M[G][H] = M[\bar{H}][G^*],$$

where $M[\bar{H}][G^*]$ is a two-stage iteration by $\mathrm{Coll}(\kappa, <\mu)$ and some quotient forcing from $M[\bar{H}]$.

Proof: Obviously, $\bar{H}$ is a filter for $(\mathrm{Coll}(\kappa, <\mu))^M$. It remains to show M-genericity. Let $\bar{D} \in M$ be dense in $\mathrm{Coll}(\kappa, <\mu)$, i.e., in M,

$$\forall p \in \mathrm{Coll}(\kappa, <\mu) \exists q \in \mathrm{Coll}(\kappa, <\mu)[q \leq p \wedge q \in \bar{D}].$$

By a reflection argument, it is possible to find $\lambda < \mu$ with $\mathrm{cof}^M(\lambda) \geq \kappa$ such that

$$\forall p \in (\mathrm{Coll}(\kappa, <\mu))^M \cap (V_\lambda)^M \exists q \in (\mathrm{Coll}(\kappa, <\mu))^M \cap (V_\lambda)^M [q \leq p \wedge q \in \bar{D}],$$

i.e., $\bar{D} \cap (V_\lambda)^M$ is dense in $(\mathrm{Coll}(\kappa, <\mu))^M \cap (V_\lambda)^M$. Define, in $M[G]$,

$$D = \{p \in (\mathrm{Coll}(\kappa, <\mu))^{M[G]} \mid p \cap (V_\lambda)^M \in \bar{D}\}.$$

We show that $D \in M[G]$ is dense in $(\mathrm{Coll}(\kappa, <\mu))^{M[G]}$. First, let $r \in (\mathrm{Coll}(\kappa, <\mu))^{M[G]}$. Then $r \cap (V_\lambda)^M \in (\mathrm{Coll}(\kappa, <\mu))^M \cap (V_\lambda)^M$, since r cannot be cofinal in λ because $\mathrm{cof}^M(\lambda) = \mathrm{cof}^{M[G]}(\lambda) \geq \kappa$. Take $q \in (\mathrm{Coll}(\kappa, <\mu))^M \cap V_\lambda$ such that $q \leq r \cap (V_\lambda)^M$ and $q \in \bar{D}$. Define $p = q \cup r \restriction (\kappa \times (\mu \setminus \lambda)) \leq r$. Then $p \in D$.

By the $M[G]$-genericity of H, take $p \in H \cap D$. Then $p \cap (V_\lambda)^M \in \bar{D}$, $p \cap (V_\lambda)^M \geq p$, $p \cap (V_\lambda)^M \in H \cap (\mathrm{Coll}(\kappa, <\mu))^M = \bar{H}$. Hence, $p \cap (V_\lambda)^M \in \bar{D} \cap \bar{H} \neq \emptyset$. This completes the proof of Lemma 10.

□

Given Lemma 10, it is now possible to prove Theorems 2 and 3. For the proof of Theorem 2, in M, let μ be measurable, and let $\mathbb{P}$ be Prikry-forcing for μ. Let G be M-generic for $\mathbb{P}$, and let H be $M[G]$-generic for $(\mathrm{Coll}(\kappa, <\mu))^{M[G]}$. Form $\bar{H}$ as above. Then $M[\bar{H}]$ is a generic extension of M in which $\mu = \kappa^+$. Thus, by Lemma 10, because $M[\bar{H}][G^*] = M[G][H]$, in $M[\bar{H}]$, there is a set generic extension preserving all cardinals $\leq \kappa$ in which κ^+ has been singularized with countable cofinality. This completes the proof of Theorem 2.

□

For the proof of Theorem 3, suppose we start with a model V of GCH in which there exist $\delta < \mu$, δ and μ regular cardinals such that $o(\mu) = \delta$. By the work of [7], V can be generically extended to a model M in which μ remains measurable, δ remains regular, and it is possible to change μ's cofinality to δ without either collapsing cardinals or adding bounded subsets of μ. Suppose that in M, κ is regular, $\delta < \kappa < \mu$. The argument used in the proof of Theorem 2 then applies to produce a generic extension $M[\bar{H}]$ having a further set generic extension in which all cardinals $\leq \kappa$ are preserved and κ^+ has been singularized with cofinality δ. This completes the proof of Theorem 3.

□

We remark that by their definitions, the partial orderings witnessing the conclusions of Theorems 2 and 3 are both $(2^\mu)^+$-c.c. In particular, assuming $2^\mu = \mu^+$, each of these forcings is μ^{++}-c.c. This is best possible, since by a theorem of Shelah (which says that if δ is a regular cardinal such that after forcing with $\mathbb{Q}$, $\mathrm{cof}(\delta) \neq |\delta|$, then $\mathbb{Q}$ collapses δ^+ — see [16, Lemma 4.9] and [10, Corollary 23.20]), both of these partial orderings must collapse μ^+.

4 Singularizing successors of singular cardinals

In this section, we will prove Theorems 5 – 8. We begin with the proof of Theorem 5.

Proof: Suppose $\kappa = \sup_{i<\omega} \kappa_i$, where $\langle \kappa_i \mid i < \omega \rangle$ is a strictly increasing sequence of κ^+-strongly compact cardinals. The definition of our forcing partial ordering $\mathbb{P}$ uses an idea of Sargsyan found in [3] which builds upon the work of [9] and [2]. Since each κ_i is κ^+-strongly compact, let $\langle \mathcal{U}_i \mid i < \omega \rangle$ be such that $\mathcal{U}_i$ is a κ_i-additive, uniform ultrafilter over κ^+. $\mathbb{P}$ may now be defined as the set of all finite sequences of the form $\langle \alpha_1, \ldots, \alpha_n, f \rangle$ satisfying the following properties.

a) $\langle \alpha_1, \ldots, \alpha_n \rangle \in [\kappa^+]^{<\omega}$.

b) f is a function having domain $T_{\alpha_1,\ldots,\alpha_n} = \{\langle \beta_1, \ldots, \beta_m \rangle \in [\kappa^+]^{<\omega} \mid \langle \alpha_1, \ldots, \alpha_n \rangle$ is an initial segment of $\langle \beta_1, \ldots, \beta_m \rangle\}$ such that $f(\langle \beta_1, \ldots, \beta_m \rangle) \in \mathcal{U}_m$.

The ordering on $\mathbb{P}$ is given by $\langle \beta_1, \ldots, \beta_m, g \rangle \le \langle \alpha_1, \ldots, \alpha_n, f \rangle$ iff the following criteria are met.

a) $\langle \alpha_1, \ldots, \alpha_n \rangle$ is an initial segment of $\langle \beta_1, \ldots, \beta_m \rangle$.

b) For $i = n+1, \ldots, m$, $\beta_i \in f(\langle \alpha_1, \ldots, \alpha_n, \ldots, \beta_{i-1} \rangle)$.

c) For every $\vec{s} \in \mathrm{dom}(g)$ (which must be a subset of $\mathrm{dom}(f)$), $g(\vec{s}) \subseteq f(\vec{s})$.

The usual density argument shows that forcing with $\mathbb{P}$ adds a cofinal ω sequence to $(\kappa^+)^V$. It is possible to prove a Prikry lemma for $\mathbb{P}$, i.e., given $\langle \alpha_1, \ldots, \alpha_n, f \rangle \in \mathbb{P}$ and formula φ in the language of forcing with respect to $\mathbb{P}$, there is a condition $\langle \alpha_1, \ldots, \alpha_n, f' \rangle \le \langle \alpha_1, \ldots, \alpha_n, f \rangle$ deciding φ. More specifically, we have the following.

Lemma 11 *Given any formula φ in the forcing language with respect to $\mathbb{P}$ and any condition $\langle \alpha_1, \ldots, \alpha_n, f \rangle \in \mathbb{P}$, there is a condition $\langle \alpha_1, \ldots, \alpha_n, f' \rangle \le \langle \alpha_1, \ldots, \alpha_n, f \rangle$ deciding φ.*

Proof: The proof of Lemma 11 is essentially the same as the proof of [3, Lemma 2] and generalizes the proofs of [9, Lemma 4.1] and [2, Lemma 1.1]. We will quote verbatim as appropriate, making the necessary minor changes where warranted. Specifically, let $s = \langle \alpha_1, \ldots, \alpha_n \rangle$, and say that $n =_{\mathrm{df}} \mathrm{length}(s)$. For any $t \in T_s$, call t sufficient if, for some g, $\langle t, g \rangle \parallel \varphi$ (i.e., $\langle t, g \rangle$ *decides* φ). For t sufficient, let g_t be a witness, with $g_t(r) = \kappa^+$ for all $r \in \mathrm{dom}(g_t)$ if t is not sufficient. If s is sufficient, then we are done. If not, then for any $t \in T_s$, sufficient or otherwise, one of the sets

$$X_t = \{\alpha < \kappa^+ \mid \exists g[\langle t^\frown\{\alpha\}, g \rangle \Vdash \varphi]\},$$

$$Y_t = \{\alpha < \kappa^+ \mid \exists g[\langle t^\frown\{\alpha\}, g \rangle \Vdash \neg\varphi]\}, \text{or}$$

$$Z_t = \{\alpha < \kappa^+ \mid \forall g[\langle t^\frown\{\alpha\}, g \rangle \text{does not decide } \varphi]\}$$

is an element of $\mathcal{U}_{\mathrm{length}(t)}$. Let A_t be that set, and for $i \le \mathrm{length}(t)$, let $t \restriction i$ be the first i members of t. For $t \in T_s$, define f' by

$$f'(t) = f(t) \cap \bigcap_{n \le i \le \mathrm{length}(t)} g_{t \restriction i}(t) \cap A_t.$$

Note that by the definition of $\mathbb{P}$, $f'(t) \in \mathcal{U}_{\mathrm{length}(t)}$, which means that $\langle s, f' \rangle$ is a well-defined member of $\mathbb{P}$ extending $\langle s, f \rangle$.

Now, let t be sufficient and of minimal length $m+1>n$, with $\langle t,f''\rangle \le \langle s,f'\rangle$ and $f''=f'\restriction T_t$. Let t' be the sequence t without its last element. It then follows that $A_{t'}$ must be either $X_{t'}$ or $Y_{t'}$, so we suppose without loss of generality that $A_{t'}=X_{t'}$. It must be the case that $\langle t',f'\restriction T_{t'}\rangle \Vdash \varphi$, since if some extension $\langle t'',g'\rangle \Vdash \neg\varphi$, such a condition must add elements to t', since t' isn't sufficient. The first element added to t', α, must come from $X_{t'}$, yielding a condition $\langle t'^\frown\{\alpha\}^\frown u,g'\rangle \Vdash \neg\varphi$. However, by construction,

$$\langle t'^\frown\{\alpha\}^\frown u,g'\rangle \le \langle t'^\frown\{\alpha\},f'\restriction T_{t'^\frown\{\alpha\}}\rangle \le \langle t'^\frown\{\alpha\},g_{t'^\frown\{\alpha\}}\rangle \Vdash \varphi,$$

which is a contradiction. Thus, $\langle t',f'\restriction T_{t'}\rangle \parallel \varphi$, which contradicts the minimality of the length of t for sufficiency. This completes the proof of Lemma 11.

$\square$

Lemma 12 *Forcing with $\mathbb{P}$ adds no new subsets of any $\delta<\kappa$.*

Proof: The proof of Lemma 12 is virtually identical to the proof of [3, Lemma 3]. As before, we will quote verbatim as appropriate, making the necessary minor changes where warranted. Given $\delta<\kappa$, suppose that $p=\langle\alpha_1,\ldots,\alpha_n,f\rangle \Vdash$ "$\tau\subseteq\delta$". Without loss of generality, by extending p if necessary, we also assume that $\kappa_n>\delta$. Further, by Lemma 11, for each $\beta<\tau$, we let $\langle\alpha_1,\ldots,\alpha_n,f_\beta\rangle$ be such that $\langle\alpha_1,\ldots,\alpha_n,f_\beta\rangle \parallel$ "$\beta\in\tau$".

Note that the domains of all of the f_β s for $\beta<\delta$ and f are the same, namely $T_{\alpha_1,\ldots,\alpha_n}$. Therefore, by the choice of p and the definition of $\mathbb{P}$, for each $s\in T_{\alpha_1,\ldots,\alpha_n}$, $f_\beta(s)$ and $f(s)$ lie in an ultrafilter $\mathcal{U}_{\text{length}(s)}$ that is κ_n-additive. This means that $g(s)=\bigcap_{\beta<\delta} f_\beta(s)\cap f(s)$ is such that $g(s)\in\mathcal{U}_{\text{length}(s)}$, and $q=\langle\alpha_1,\ldots,\alpha_n,g\rangle$ is a well-defined element of $\mathbb{P}$ such that $q\le p$ and q decides the statement "$\beta\in\tau$" for every $\beta<\delta$. Hence, forcing with $\mathbb{P}$ adds no new subsets of δ. This completes the proof of Lemma 12.

$\square$

By Lemma 12, forcing with $\mathbb{P}$ adds no new bounded subsets of κ. Since any two conditions having the same stem are compatible and the number of stems is $|[\kappa^+]^{<\omega}|=\kappa^+$, $\mathbb{P}$ is κ^{++}-c.c. This completes the proof of Theorem 5.

$\square$

To prove Theorem 6, suppose $V\vDash$ "κ is κ^+-strongly compact". Fix $\mathcal{U}$ a κ-additive, fine measure over $P_\kappa(\kappa^+)$. We may now define strongly compact Prikry forcing $\mathbb{Q}$ as in [2] as the set of all finite sequences of the form $\langle p_1,\ldots,p_n,f\rangle$ satisfying the following properties.

a) Each $p_i\in P_\kappa(\kappa^+)$.

b) $p_1\subseteq\cdots\subseteq p_n$.

c) f is a function having domain $T_{p_1,\ldots,p_n}=\{\langle q_1,\ldots,q_m\rangle \mid q_1\subseteq\cdots\subseteq q_m$ and $\langle p_1,\ldots,p_n\rangle$ is an initial segment of $\langle q_1,\ldots,q_m\rangle\}$ such that $f(\langle q_1,\ldots,q_m\rangle)\in\mathcal{U}$.

The ordering on $\mathbb{P}$ is given by $\langle q_1,\ldots,q_m,g\rangle \le \langle p_1,\ldots,p_n,f\rangle$ iff the following criteria are met.

a) $\langle p_1,\ldots,p_n\rangle$ is an initial segment of $\langle q_1,\ldots,q_m\rangle$.

b) For $i=n+1,\ldots,m$, $q_i\in f(\langle p_1,\ldots,p_n,\ldots,q_{i-1}\rangle)$.

c) For every $\vec{s} \in \mathrm{dom}(g)$ (which must be a subset of $\mathrm{dom}(f)$), $g(\vec{s}) \subseteq f(\vec{s})$.

Let G be V-generic over $\mathbb{Q}$. Because $|P_\kappa(\kappa^+)| = |[\kappa^+]^{<\kappa}| = \kappa^+$ and $|[\kappa^+]^{<\omega}| = \kappa^+$, there are only κ^+ many possibilities for stems for members of $\mathbb{Q}$. Since any two conditions having the same stem are compatible, $\mathbb{Q}$ is therefore κ^{++}-c.c. By [2, Lemma 1.1] and the remark in the paragraph immediately following, forcing with $\mathbb{Q}$ adds no new bounded subsets of κ. Thus, κ is a cardinal in $V[G]$. A routine density argument (mentioned in the paragraph immediately prior to [2, Lemma 1.1]) tells us that the ω sequence $r = \langle p_i \mid i < \omega \rangle$ generated by G changes the cofinality of both κ and $(\kappa^+)^V$ to ω and also collapses $(\kappa^+)^V$ to κ. And, if we let $r \restriction \kappa = \langle p_i \cap \kappa \mid i < \omega \rangle$, by the density argument just mentioned, $r \restriction \kappa$ also changes κ's cofinality to ω. Consequently, in $V[r \restriction \kappa] \subseteq V[G]$, κ is a singular cardinal having cofinality ω.

For $A \in \mathcal{U}$, let $A \restriction \kappa = \{p \cap \kappa \mid p \in A\}$. Define $\mathcal{U} \restriction \kappa = \{A \restriction \kappa \mid A \in \mathcal{U}\}$. It is easy to verify that $\mathcal{U} \restriction \kappa$ is a κ-additive, fine measure over $P_\kappa(\kappa)$. Let $\mathbb{Q}_{\mathcal{U}\restriction\kappa}$ be defined in the same manner as $\mathbb{Q}$ except that $\mathcal{U} \restriction \kappa$ is used in its definition instead of $\mathcal{U}$. Because $|P_\kappa(\kappa)| = |[\kappa]^{<\kappa}| = \kappa$ and $|[\kappa]^{<\omega}| = \kappa$, there are only κ many possibilities for stems for members of $\mathbb{Q}_{\mathcal{U}\restriction\kappa}$. As in the preceding paragraph, this just means that $\mathbb{Q}_{\mathcal{U}\restriction\kappa}$ is κ^+-c.c. Further, since [2, Lemma 1.5] and the paragraph immediately following tell us that $r \restriction \kappa$ generates a V-generic object G^* for $\mathbb{Q}_{\mathcal{U}\restriction\kappa}$ and r generates G, $V[r \restriction \kappa] = V[G^*]$ and $V[r] = V[G]$. Hence, $(\kappa^+)^{V[r\restriction\kappa]} = (\kappa^+)^{V[G^*]} = (\kappa^+)^V$. In addition, as $V[r \restriction \kappa] \subseteq V[r]$, V, $V[r \restriction \kappa]$, and $V[r]$ all contain the same bounded subsets of κ.

Working now in $V[r \restriction \kappa]$, let $\mathbb{P} = \mathbb{Q}/(r \restriction \kappa)$, i.e., $\mathbb{P}$ is the quotient forcing of $\mathbb{Q}$ with respect to $r \restriction \kappa$. Take G^{**} as the quotient generic object G/G^*. $\mathbb{P}$ is the desired forcing over $V[r \restriction \kappa]$ witnessing $\mathrm{Sing}(\kappa^+, \aleph_0)$. This is since $V[r \restriction \kappa]$ and $V[r \restriction \kappa][G^{**}] = V[G^*][G^{**}] = V[G]$ contain the same bounded subsets of κ, $\mathrm{cof}^{V[G]}((\kappa^+)^{V[r\restriction\kappa]}) = \omega$, and $\mathbb{P} = \mathbb{Q}/(r \restriction \kappa)$ is κ^{++}-c.c. because $\mathbb{Q}$ is. This completes the proof of Theorem 6.

$\square$

When proving Theorems 7 and 8, we will omit precise definitions of the partial orderings, generic extensions, and submodels used, which are rather technical in nature. Instead, we refer readers to the relevant paper by Magidor. In particular, for the proof of Theorem 7, suppose $V \models$"ZFC + κ is κ^+-supercompact". By the proof of [13, Theorem 1], there is a partial ordering $\mathbb{Q} \in V$ and submodel $V' \subseteq V[G]$ (where G is V-generic over $\mathbb{Q}$) such that the following hold.

a) G generates generic sequences $r = \langle p_i \mid i < \omega \rangle$ and $\vec{f} = \langle f_i \mid i < \omega \rangle$ such that r changes the cofinality of both κ and κ^+ to $\aleph_0$ and $\vec{f}$ collapses κ to $\aleph_\omega$.

b) $V' = V[\langle r \restriction \kappa, \vec{f} \rangle]$, where as in the proof of Theorem 6, $r \restriction \kappa = \langle p_i \cap \kappa \mid i < \omega \rangle$.

c) In both V' and $V[G]$, $\kappa = \aleph_\omega$.

d) V' and $V[G]$ contain the same bounded subsets of κ.

e) $(\kappa^+)^V = (\kappa^+)^{V'} < (\kappa^+)^{V[G]}$.

f) $\mathrm{cof}^{V[G]}((\kappa^+)^{V'}) = \aleph_0$.

g) V' and $V[G]$ have the same cardinals $\geq (\kappa^{++})^V$.

If we now let $\mathbb{Q}^*$ be the partial ordering and G^* the V-generic object over $\mathbb{Q}^*$ such that $V[G^*] = V'$, take $\mathbb{P} = \mathbb{Q}/(\langle r \restriction \kappa, \vec{f} \rangle)$ as the quotient forcing of $\mathbb{P}$ with respect to $\langle r \restriction \kappa, \vec{f} \rangle$, and let G^{**} once again be the quotient generic object G/G^*, then the argument found in the last paragraph of the proof of Theorem 6 in tandem with properties a) – g) above show that $\mathbb{P}$ is the desired forcing witnessing $\mathrm{Sing}(\aleph_{\omega+1}, \aleph_0)$ over V'. This completes the proof of Theorem 7.

$\square$

Finally, to prove Theorem 8, suppose $V \vDash$"ZFC + κ is κ^{++}-supercompact + $2^{\kappa^+} = \kappa^{++}$". By the proof of [13, Theorem 2], there is a partial ordering $\mathbb{Q} \in V$ and submodel $V' \subseteq V[G]$ (where G is V-generic over $\mathbb{Q}$) such that the following hold.

a) G generates generic sequences $r = \langle p_i \mid i < \omega_1 \rangle$ and $\vec{f} = \langle f_i \mid i < \omega_1 \rangle$ such that r changes the cofinality of both κ and κ^+ to $\aleph_1$ and $\vec{f}$ collapses κ to $\aleph_{\omega_1}$.

b) $V' = V[\langle r \restriction \kappa, \vec{f} \rangle]$, where in analogy to the proofs of Theorems 6 and 7, $r \restriction \kappa = \langle p_i \cap \kappa \mid i < \omega_1 \rangle$.

c) In both V' and $V[G]$, $\kappa = \aleph_{\omega_1}$.

d) $(\kappa^+)^V = (\kappa^+)^{V'} < (\kappa^+)^{V[G]}$.

e) $\mathrm{cof}^{V[G]}((\kappa^+)^{V'}) = \aleph_1$.

f) V' and $V[G]$ have the same cardinals $\geq (\kappa^{++})^V$.

If we once again let $\mathbb{Q}^*$ be the partial ordering and G^* the V-generic object over $\mathbb{Q}^*$ such that $V[G^*] = V'$, take $\mathbb{P} = \mathbb{Q}/(\langle r \restriction \kappa, \vec{f} \rangle)$ as the quotient forcing of $\mathbb{P}$ with respect to $\langle r \restriction \kappa, \vec{f} \rangle$, and let G^{**} be the quotient generic object G/G^*, then the arguments found above in tandem with properties a) – f) show that $\mathbb{P}$ is the desired forcing over V' singularizing $\aleph_{\omega_1+1}$ in cofinality $\aleph_1$ while preserving $\aleph_{\omega_1}$ and also preserving all cardinals $\geq \aleph_{\omega_1+2}$. This completes the proof of Theorem 8.

$\square$

5 Remarks and further questions

We note that there are key differences between the singularizing forcings constructed in Theorems 7 and 8. Specifically, the partial ordering $\mathbb{P}$ of Theorem 7 will not add new bounded subsets of $\aleph_\omega$ (and therefore won't collapse any cardinals below $\aleph_\omega$), whereas the partial ordering $\mathbb{P}$ of Theorem 8 will collapse cardinals below $\aleph_{\omega_1}$ (and thus adds new bounded subsets of $\aleph_{\omega_1}$). This is since the sequence $\langle p_i \mid i < \omega_1 \rangle$ collapses cardinals below κ (see [13] for further details). Also, easy modifications of the definitions of the singularizing forcings found in Theorems 7 and 8 will allow us to prove analogues of these theorems for other "small" singular cardinals of both countable and uncountable cofinality, as well as an analogue of Theorem 6 for uncountable cofinality (using supercompactness instead of strong compactness assumptions).

The theorems in this paper raise a number of questions. In particular:

1. Can we have sequences of consecutive successor cardinals below $\aleph_\omega$ which are simultaneously singularizable? This is possible for $\aleph_2$ (since Namba forcing is always definable in any model of ZFC) and $\aleph_3$ (by Theorems 2 and 3 above). Are there other examples, of length 2 or longer? Note that the existence of such sequences has high consistency strength. In particular, if κ and κ^+ are both singularizable with $\kappa > \aleph_2$, then there is an inner model with a Woodin cardinal.[1]

2. Can we obtain a model of ZFC in which κ is inaccessible and it is possible to singularize all regular cardinals in the interval $[\aleph_2, \kappa)$ while preserving that κ is a cardinal?

3. Are there other successors of singular cardinals at which one can outright prove that a singularizing forcing exists, assuming the appropriate large cardinal hypotheses?

4. Is it possible to weaken the large cardinal assumptions used to prove Theorems 5 – 8? What are the optimal hypotheses?

5. Is it possible to prove versions of Theorems 5 – 8 in which the cofinality of κ^+, the successor of the singular cardinal, is changed to a cofinality different from κ's? This can be done with the stationary tower forcing. Note that by Shelah's theorem of [16, Lemma 4.9] and [10, Corollary 23.20], any partial ordering accomplishing this would have to collapse (at least) κ^{++} as well as κ^+.

6. Is it possible to prove a version of Theorem 8 in which no bounded subsets of $\aleph_{\omega_1}$ are added by the singularizing forcing? More weakly, is it possible to prove a version of Theorem 8 in which no cardinals below $\aleph_{\omega_1}$ are collapsed? The analogous stationary tower forcing will not add bounded subsets of $\aleph_{\omega_1}$ assuming GCH. Work of Gitik [8] shows that any partial ordering with a definition similar to the one given in Theorem 8 will of necessity have to collapse a stationary subset of cardinals below $\aleph_{\omega_1}$.

7. What applications (if any) are there for the singularizing forcings constructed in this paper?

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