CHARACTERIZATION OF THE METAL-INSULATOR TRANSPORT TRANSITION FOR THE TWO-PARTICLE ANDERSON MODEL

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Abstract. We extend to the two-particle Anderson model the characterization of the metal-insulator transport transition obtained in the one-particle setting by Germinet and Klein. We show that, for any fixed number of particles, the slow spreading of wave packets in time implies the initial estimate of a modified version of the Bootstrap Multiscale Analysis. In this new version, operators are restricted to boxes defined with respect to the pseudo-distance in which we have the slow spreading. At the bottom of the spectrum, within the regime of one-particle dynamical localization, we show that this modified multiscale analysis yields dynamical localization for the two-particle Anderson model, allowing us to obtain a characterization of the metal-insulator transport transition for the two-particle Anderson model at the bottom of the spectrum.

Contents

1. Introduction

Localization, or absence of transport, is considered to be a characteristic feature of random media. In the well known one-particle Anderson model, it is known to appear at either high disorder or low energies. In recent years the multi-particle

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Anderson model has attracted great interest, as it is expected that localization persists in the presence of inter-particle interactions. This has been shown to be the case for short-range interactions, using either a multiscale analysis (MSA) or the fractional moment method [ChS1, ChS2, ChS3, AW1, BCSS, ChBS, KlN1, KlN2]. More recently, interactions of exponential decay and even fast polynomial decay have been considered in [FW].

In this paper we contribute to the efforts to understand the regime of localization in the multi-particle setting by extending the work of Germinet and Klein [GK4] on the characterization of the metal-insulator transport transition for the Anderson model. In the aforementioned work, the authors used transport exponents to measure the spreading of wave packets, spliting the spectrum of the operator into two complementary regions: the *metallic region*, where non trivial transport occurs, and the *insulator region*, where transport is suppressed by dynamical localization. Germinet and Klein showed that the insulator region, where the transport exponent is null, is equivalent to the set of energies where the MSA can be applied, making this method their main tool of analysis. Moreover, they gave a lower bound for the transport exponent in the metallic region, which implies that the mobility edge, i.e., the energy that separates the regions of localization and delocalization, is a point of discontinuity for the transport exponent.

In our work we follow the approach of [GK4], where the proof consists of two parts: on one hand, slow transport, identified as slow growth of the time-averaged spreading of wave packets, implies the starting hypothesis of the MSA; on the other hand, the MSA implies dynamical localization and therefore null transport. Therefore, if there is spreading of wave packets, this must be at a rate that is above a minimal amount. In the one-particle setting, this characterization proved to be crucial in the study of the Landau Hamiltonian perturbed by a random potential [GKS], one of the few models where the Anderson transport transition has been rigorously proved, together with the Anderson model in the Bethe lattice [Kl1, Kl2, ASW, FrHS, AW2].

In the multi-particle setting dynamical localization is obtained with respect to the Hausdorff pseudo-distance in the multi-particle space, but the MSA (or fractional moment method) is based on the usual boxes in the multi-particle space, i.e., boxes defined by the norm given by the maximum of the absolute value of the coordinates [ChS1, ChS2, ChS3, AW1, BCSS, ChBS, KlN1, KlN2]. The initial step for the MSA is a statement about the finite-volume restrictions of the random Schrödinger operators to usual boxes, but the statement of dynamical localization is only with respect to the Hausdorff pseudo-distance. For two particles, i.e., $N = 2$, the Hausdorff and symmetrized pseudo-distances are the same, so the statement of dynamical localization can be seen as being with respect to the symmetrized pseudo-distance.

In order to extend the characterization of Germinet and Klein to the multi-particle setting, we first show that the slow spreading of the N-particle wave-packets for large times, with respect to either the Hausdorff or symmetrized pseudo-distance, implies the initial step of a modified MSA in which the finite-volume restrictions of operators are defined using boxes with respect to the relevant pseudo-distance. This holds for any fixed number of particles N, with either the Hausdorff or symmetrized pseudo-distance, since the Wegner estimate holds on boxes defined by either pseudo-distance.

To obtain the characterization of the metal-insulator transition as in [GK4], we need to perform a variant of the Bootstrap MSA (see [KlN1] for the Bootstrap MSA for multi-particle Anderson models) using boxes defined by the pseudo-metric. The deterministic part of the MSA can be done for boxes defined by the symmetrized pseudo-distance, but the probabilistic part of this modified MSA would require a Wegner estimate between boxes that are far apart in the symmetrized pseudo-distance. For the multi-particle Anderson model where the single-site probability distribution is only assumed to have a bounded density with compact support such a Wegner estimate is only known between boxes that are far apart in the Hausdorff distance, and hence we only have it for the symmetrized distance if $N = 2$. For this reason we restrict ourselves to two particles, for which we prove dynamical localization with respect to the symmetrized pseudo-distance, with the initial step for the MSA being a statement about the finite-volume restrictions of the random operators to symmetrized boxes. Thus for two particles we obtain a characterization of the metal-insulator transition in the spirit of [GK4] for quantities defined with respect to the symmetrized pseudo-distance.

Recently Chulaevsky obtained a Wegner estimate between boxes that are far apart in the symmetrized distance for all N, for a certain class of single site probability distributions [ChS4] (see also [Ch1]). Under Chulaevsky's assumptions our modified Bootstrap MSA for boxes in the symmetrized pseudo-distance can be performed for N-particles, where N is fixed but arbitrary, and our characterization of the metal-insulator transition holds for N-particles.

The article is organized as follows: in the next section we set the notation and state the main results. In Section 3 we prove that slow transport implies the initial step of a modified Bootstrap MSA. Section 4 is devoted to the Bootstrap Multiscale Analysis for symmetrized two-particle boxes. In Subsection 4.1 we explain the modifications to the N-particle MSA of [KlN1, KlN2] that are necessary in our setting, and give details on how this modifications are implemented in Subsection 4.2. In Section 5 we show that whenever this MSA can be performed in an N-particle setting, it yields dynamical localization with respect to the symmetrized pseudo-distance, which completes the proof of the main result. At the same time, we are able to improve the conclusions of [KlN1, Corollary 1.7] and [KlN2, Theorem 1.2] by removing the dependency on the initial position of the particles in the statement of dynamical localization. Appendix A discusses Wegner estimates for multi-particle rectangles and boxes defined by the relevant pseudo-distances. In Appendix B we state and prove a Combes-Thomas estimate for restrictions of discrete Schrödinger operators to arbitrary subsets. In Appendix C we comment on auxiliary results known in the one-particle case, that are generalized to the multi-particle setting and an arbitrary pseudo-distance.

2. Main definitions and results

We start by defining the n-particle Anderson model.

Definition 2.1. *The n-particle Anderson model is the random Schrödinger operator on $\ell^2(\mathbb{Z}^{nd})$ given by*

$$H_{\boldsymbol{\omega}}^{(n)} := -\Delta^{(n)} + V_{\boldsymbol{\omega}}^{(n)} + U, \tag{2.1}$$

where:

(i) $\Delta^{(n)}$ *is the discrete nd-dimensional Laplacian operator.*

(ii) $\boldsymbol{\omega} = \{\omega_x\}_{x\in\mathbb{Z}^d}$ *is a family of independent identically distributed random variables whose common probability distribution* μ *has a bounded density* ρ *and satisfies* $\{0, M_+\} \subset \operatorname{supp}\mu \subseteq [0, M_+]$ *for some* $M_+ > 0$.

(iii) $V_{\boldsymbol{\omega}}^{(n)}$ *is the random potential given by*

$$V_{\boldsymbol{\omega}}^{(n)}(\mathbf{x}) = \sum_{i=1,\dots,n} V_{\boldsymbol{\omega}}^{(1)}(x_i) \quad \textit{for} \quad \mathbf{x} = (x_1, ..., x_n) \in \mathbb{Z}^{nd}, \tag{2.2}$$

where $V_{\boldsymbol{\omega}}^{(1)}(x) = \omega_x$ *for every* $x \in \mathbb{Z}^d$.

(iv) U *is a potential governing the short range interaction between the* n *particles. We take*

$$U(\mathbf{x}) = \sum_{1\le i<j\le n} \widetilde{U}(x_i - x_j), \tag{2.3}$$

where $\widetilde{U}\colon \mathbb{Z}^d \to [0,\infty)$, $\widetilde{U}(y) = \widetilde{U}(-y)$, *and* $\widetilde{U}(y) = 0$ *for* $\|y\|_\infty > r_0$ *for some* $r_0 \in [1,\infty)$.

The n-particle Anderson model is a $\mathbb{Z}^d$-ergodic random Schrödinger operator. It follows from standard arguments (cf. [CL, Proposition V.2.4]) that there exists a bounded set $\Sigma^{(n)} \subset \mathbb{R}$ such that $\sigma(H_{\boldsymbol{\omega}}^{(n)}) = \Sigma^{(n)}$ almost-surely, where $\sigma(H)$ denotes the spectrum of the operator H. Since $\sigma(-\Delta^{(n)}) = [0, 4nd]$, and both $V_{\boldsymbol{\omega}}^{(n)}$ and U are non-negative, we have $\Sigma^{(n)} \subset [0, +\infty)$.

We will generally omit $\boldsymbol{\omega}$ from the notation, and use the following notation and definitions:

(i) Given $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we set $\|x\| = \|x\|_\infty := \max\{|x_1|, \dots, |x_d|\}$. If $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^{nd}$, we let $\|\mathbf{a}\| := \max\{\|a_1\|, \dots, \|a_n\|\}$, $\mathcal{S}_{\mathbf{a}} = \{a_1, ..., a_n\}$, and $\langle \mathbf{a}\rangle := \langle \|\mathbf{a}\|\rangle$, where $\langle t\rangle := \sqrt{1+t^2}$ for $t \ge 0$.

(ii) For $n \in \mathbb{N}$, we set $\mathcal{P}_n$ to be the set of all permutations of n elements. Moreover, if $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^{nd}$ and $\pi \in \mathcal{P}_n$, we write $\pi(\mathbf{x}) = \big(x_{\pi(1)}, \dots, x_{\pi(n)}\big)$.

(iii) We introduce one distance and two pseudo-distances (which we will incorrectly call distances) in $\mathbb{R}^{nd}$. They are defined as follows for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{nd}$:

(a) The norm distance: $\operatorname{dist}_\infty(\mathbf{a}, \mathbf{b}) = \operatorname{dist}(\mathbf{a}, \mathbf{b}) := \|\mathbf{a} - \mathbf{b}\|$.

(b) The symmetrized distance:

$$\operatorname{dist}_S(\mathbf{a}, \mathbf{b}) := \min_{\pi\in\mathcal{S}_n} \{\|\pi(\mathbf{a}) - \mathbf{b}\|\} = \min_{\pi\in\mathcal{S}_n} \{\|\pi(\mathbf{b}) - \mathbf{a}\|\}. \tag{2.4}$$

(c) The Hausdorff distance:

$$\begin{aligned} \operatorname{dist}_H(\mathbf{a}, \mathbf{b}) &:= \operatorname{dist}_H(\mathcal{S}_{\mathbf{a}}, \mathcal{S}_{\mathbf{b}}) \\ &:= \max\Big\{\max_{x\in\mathcal{S}_{\mathbf{a}}} \min_{y\in\mathcal{S}_{\mathbf{b}}} \|x-y\|\,,\, \max_{y\in\mathcal{S}_{\mathbf{b}}} \min_{x\in\mathcal{S}_{\mathbf{a}}} \|x-y\|\Big\} \\ &= \max\Big\{\max_{x\in\mathcal{S}_{\mathbf{a}}} \operatorname{dist}(x,\ \mathcal{S}_{\mathbf{b}})\,,\, \max_{y\in\mathcal{S}_{\mathbf{b}}} \operatorname{dist}(y,\ \mathcal{S}_{\mathbf{a}})\Big\}. \end{aligned} \tag{2.5}$$

We will write $\operatorname{dist}_\sharp(\mathbf{a}, \mathbf{b})$, where $\sharp \in \{\infty, S, H\}$, to denote either one of these distances, as appropriate. Note that

$$\operatorname{dist}_H(\mathbf{a}, \mathbf{b}) \le \operatorname{dist}_S(\mathbf{a}, \mathbf{b}) \le \operatorname{dist}_\infty(\mathbf{a}, \mathbf{b}). \tag{2.6}$$

If $n = 2$, we have $\operatorname{dist}_H(\mathbf{a}, \mathbf{b}) = \operatorname{dist}_S(\mathbf{a}, \mathbf{b})$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{2d}$.

(iv) The n-particle $\sharp$-box of side $L \geq 1$, where $\sharp \in \{\infty, S, H\}$, centered at $\mathbf{x} \in \mathbb{R}^{nd}$, is defined by

$$\mathbf{\Lambda}^{(n)}_{\sharp;L}(\mathbf{x}) = \left\{\mathbf{y} \in \mathbb{Z}^{nd};\ \operatorname{dist}_\sharp(\mathbf{x},\mathbf{y}) \leq \tfrac{L}{2}\right\}. \tag{2.7}$$

$\mathbf{\Lambda}^{(n)}_{\infty;L}(\mathbf{x}) = \mathbf{\Lambda}^{(n)}_{L}(\mathbf{x})$ is the n-particle box of side L centered at $\mathbf{x}$, $\mathbf{\Lambda}^{(n)}_{S;L}(\mathbf{x})$ is the symmetrized n-particle box of side L, and $\mathbf{\Lambda}^{(n)}_{H;L}(\mathbf{x})$ is the Hausdorff n-particle box of side L. Note that

$$\mathbf{\Lambda}^{(n)}_{S;L}(\mathbf{x}) = \bigcup_{\pi \in \mathcal{P}_n} \mathbf{\Lambda}_L(\pi(\mathbf{x})) \subset \mathbf{\Lambda}^{(n)}_{H;L}(\mathbf{x}) \subset \bigcup_{\mathbf{y} \in \mathcal{S}^n_{\mathbf{x}}} \mathbf{\Lambda}_L(\mathbf{y}). \tag{2.8}$$

We also define n-particle $\sharp$-rectangles for $\sharp \in \{\infty, S\}$, centered at $\mathbf{x} \in \mathbb{R}^{nd}$ with sides $\boldsymbol{L} = (L_1, L_2, \ldots, L_n) \in [1,\infty)^n$, by

$$\mathbf{\Lambda}^{(n)}_{\boldsymbol{L}}(\mathbf{x}) = \prod_{j=1}^{n} \Lambda_{L_j}(x_j), \tag{2.9}$$

$$\mathbf{\Lambda}^{(n)}_{S;\boldsymbol{L}}(\mathbf{x}) = \bigcup_{\pi \in \mathcal{P}_n} \pi\left(\mathbf{\Lambda}^{(n)}_{\boldsymbol{L}}(\mathbf{x})\right) = \bigcup_{\pi \in \mathcal{P}_n} \prod_{j=1}^{n} \Lambda_{L_{\pi(j)}}(x_{\pi(j)}).$$

(v) We denote by $\{\delta_{\mathbf{x}};\ \mathbf{x} \in \mathbb{Z}^{nd}\}$ the canonical orthonormal base of $\ell^2(\mathbb{Z}^{nd})$.

(vi) Given $\mathbf{\Lambda}_1 \subset \mathbf{\Lambda}_2 \subset \mathbb{Z}^{nd}$, we set

$$\partial^{\mathbf{\Lambda}_2}\mathbf{\Lambda}_1 = \{(\mathbf{u},\mathbf{v}) \in \mathbf{\Lambda}_1 \times (\mathbf{\Lambda}_2 \setminus \mathbf{\Lambda}_1)\,;\, \|\mathbf{u}-\mathbf{v}\|_1 = 1\},$$

where $\|\cdot\|_1$ is the graph-norm, (i.e., the 1-norm) in $\mathbb{Z}^{nd}$,

$$\partial^{\mathbf{\Lambda}_2}_{+}\mathbf{\Lambda}_1 = \{\mathbf{v} \in \mathbf{\Lambda}_2 \setminus \mathbf{\Lambda}_1\,;\, (\mathbf{u},\mathbf{v}) \in \partial^{\mathbf{\Lambda}_2}\mathbf{\Lambda}_1 \text{ for some } \mathbf{u} \in \mathbf{\Lambda}_1\},$$

$$\partial^{\mathbf{\Lambda}_2}_{-}\mathbf{\Lambda}_1 = \{\mathbf{u} \in \mathbf{\Lambda}_1\,;\, (\mathbf{u},\mathbf{v}) \in \partial^{\mathbf{\Lambda}_2}\mathbf{\Lambda}_1 \text{ for some } \mathbf{v} \in \mathbf{\Lambda}_2 \setminus \mathbf{\Lambda}_1\}.$$

If $\mathbf{\Lambda}_2 = \mathbb{Z}^{nd}$, it may be ommitted from the notation.

(vii) Given $\mathbf{\Theta} \subset \mathbb{Z}^{nd}$, we define the finite-volume operator $H_{\mathbf{\Theta}}$ as the self-adjoint operator $H_{\mathbf{\Theta}} = \chi_{\mathbf{\Theta}} H^{(n)} \chi_{\mathbf{\Theta}}$ on $\ell^2(\mathbf{\Theta})$ obtained by restricting $H^{(n)}$ to $\mathbf{\Theta}$ with Dirichlet (simple) boundary condition. If $z \notin \sigma(H_{\mathbf{\Theta}})$, we set $G_{\mathbf{\Theta}}(z) = (H_{\mathbf{\Theta}} - z)^{-1}$ and $G_{\mathbf{\Theta}}(z,\mathbf{u},\mathbf{y}) = \langle \delta_{\mathbf{u}}, (H_{\mathbf{\Theta}} - z)^{-1}\delta_{\mathbf{y}}\rangle$ for $\mathbf{u},\mathbf{y} \in \mathbf{\Theta}$.

(viii) Given an open interval $I \subseteq \mathbb{R}$, we denote $C^\infty_c(I)$ to be the class of real-valued infinitely differentiable functions on $\mathbb{R}$ with compact support contained in I, with $C^\infty_{c,+}(I)$ being the subclass of nonnegative functions.

The random $\sharp$-moment of order $p \geq 0$ at time t for the time evolution, initially spatially localized in the n-particle box of side one around the point $\mathbf{y} \in \mathbb{Z}^{nd}$, and localized in energy by a function $g \in C^\infty_{c,+}(\mathbb{R})$, is given by

$$M^{(n,\sharp)}_{\boldsymbol{\omega}}(p,g,t,\mathbf{y}) = \sum_{\mathbf{u} \in \mathbb{Z}^{nd}} \langle \operatorname{dist}_\sharp(\mathbf{y},\mathbf{u})\rangle^p \left|\left\langle \delta_{\mathbf{u}}, e^{-itH^{(n)}_{\boldsymbol{\omega}}} g\left(H^{(n)}_{\boldsymbol{\omega}}\right)\delta_{\mathbf{y}}\right\rangle\right|^2. \tag{2.10}$$

The expectation of the random $\sharp$-moment is given by

$$\mathbf{M}^{(n,\sharp)}(p,g,t,\mathbf{y}) = \mathbb{E}\left(M^{(n,\sharp)}_{\boldsymbol{\omega}}\right), \tag{2.11}$$

and its time-averaged expectation is defined as

$$\mathcal{M}^{(n,\sharp)}(p,g,T,\mathbf{y}) = \frac{2}{T}\int_0^\infty e^{-\frac{2t}{T}}\mathbf{M}^{(n,\sharp)}(p,g,t,\mathbf{y})\,dt, \tag{2.12}$$

Remark 2.2. *Note that both* $\boldsymbol{M}^{(n,\sharp)}(p,g,t,\mathbf{y})$ *and* $\mathcal{M}^{(n,\sharp)}(p,g,T,\mathbf{y})$ *are invariant under the action of* $\mathbb{Z}^d$*, that is, for all* $a \in \mathbb{Z}^d$ *we have* $\boldsymbol{M}^{(n,\sharp)}(p,g,t,\mathbf{y}) = \boldsymbol{M}^{(n,\sharp)}(p,g,t,\tau_a(\mathbf{y}))$ *and* $\mathcal{M}^{(n,\sharp)}(p,g,T,\mathbf{y}) = \mathcal{M}^{(n,\sharp)}(p,g,T,\tau_a(\mathbf{y}))$*, where* $\tau_a(\mathbf{y}) = (y_1 - a, .., y_n - a)$*.*

For $p \geq 0$ and non-zero $g \in C^\infty_{c,+}(\mathbb{R})$, we consider the upper and lower transport exponents, defined by

$$\beta^+_{(n,\sharp)}(p,g) := \limsup_{T\to\infty} \frac{\log \sup_{\mathbf{y}\in\mathbb{Z}^{nd}} \mathcal{M}^{(n,\sharp)}(p,g,T,\mathbf{y})}{p\log T}, \tag{2.13}$$
$$\beta^-_{(n,\sharp)}(p,g) := \liminf_{T\to\infty} \frac{\log \sup_{\mathbf{y}\in\mathbb{Z}^{nd}} \mathcal{M}^{(n,\sharp)}(p,g,T,\mathbf{y})}{p\log T}.$$

Note that these quantities are well defined, that is, $\sup_{\mathbf{y}\in\mathbb{Z}^{nd}} \mathcal{M}^{(n,\sharp)}(p,g,T,\mathbf{y}) > 0$, see Remark 2.5. If $g(H^{(n)}_{\boldsymbol{\omega}}) = 0$ almost surely, we set $\beta^{\pm}_{(n,\sharp)}(p,g) = 0$.

Following [GK4, Eq. 2.16 - 2.20], we define the p-th upper and lower transport exponents in an open interval I, and the p-th local upper and lower transport exponents at an energy $E \in \mathbb{R}$, by

$$\beta^{\pm}_{(n,\sharp)}(p,I) = \sup_{g\in C^\infty_{c,+}(I)} \beta^{\pm}_{(n,\sharp)}(p,g), \tag{2.14}$$
$$\beta^{\pm}_{(n,\sharp)}(p,E) = \inf_{I\ni E,\ I \text{ open interval}} \beta^{\pm}_{(n,\sharp)}(p,I).$$

The asymptotic upper and lower transport exponents and the local asymptotic transport exponents are defined in the same way as in [GK4, Eq. 2.16-2.20], using the fact that the exponents defined above are non-decreasing in p (see Proposition 2.4),

$$\beta^{\pm}_{(n,\sharp)}(I) = \lim_{p\to\infty} \beta^{\pm}_{(n,\sharp)}(p,I) = \sup_p \beta^{\pm}_{(n,\sharp)}(p,I), \tag{2.15}$$
$$\beta^{\pm}_{(n,\sharp)}(E) = \lim_{p\to\infty} \beta^{\pm}_{(n,\sharp)}(p,E) = \sup_p \beta^{\pm}_{(n,\sharp)}(p,I).$$

In the following, we list properties of the (random) moments and their consequences for the transport exponents. These statements are proven similarly to their continuous analogs, proven in [GK4].

Proposition 2.3. *Let* $g \in C^\infty_{c,+}(\mathbb{R})$*,* $g(H^{(n)}_\omega) \neq 0$ *with probability one. We have, for every* $\mathbf{y} \in \mathbb{Z}^{nd}$ *and* $p \geq 0$*,*

$$0 \leq M^{(n,\sharp)}_\omega(0,g,0,\mathbf{y}) \leq M^{(n,\sharp)}_\omega(p,g,t,\mathbf{y}) \leq C_{d,g,p}\langle t\rangle^{\lfloor p+nd\rfloor+2} \tag{2.16}$$
$$0 \leq \mathbf{M}^{(n,\sharp)}(0,g,0,\mathbf{y}) \leq \mathbf{M}^{(n,\sharp)}(p,g,t,\mathbf{y}) \leq C_{d,g,p}\langle t\rangle^{\lfloor p+nd\rfloor+2} \tag{2.17}$$
$$0 \leq \mathbf{M}^{(n,\sharp)}(0,g,0,\mathbf{y}) \leq \mathcal{M}^{(n,\sharp)}(p,g,T,\mathbf{y}) \leq C_{d,g,p}\langle T\rangle^{\lfloor p+nd\rfloor+2}, \tag{2.18}$$

where $\lfloor x\rfloor$ *denotes the largest integer less than or equal to* $x \in \mathbb{R}$*.*

We have [GK4, Proposition 3.2],

Proposition 2.4. *Let* $g \in C^\infty_{c,+}(\mathbb{R})$ *and* $E \in \mathbb{R}$*, then*

(i) $\beta^{\pm}_{(n,\sharp)}(p,g)$ *is monotone increasing in* $p \geq 0$*.*
(ii) $0 \leq \beta^{\pm}_{(n,\sharp)}(p,g) \leq 1$*.*

Remark 2.5. *For simplicity, we write $g(H^{(n)}_{\omega}) = g(H)$. Let us look at the quantity*

$$\mathbb{E}\left(\|g(H)\delta_{\mathbf{u}}\|^2\right) = \mathbf{M}^{(n,\sharp)}(0,g,0,\mathbf{u})$$

Since $\{\delta_{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{Z}^{nd}}$ is an orthonormal base, we have that $\mathbf{M}^{(n,\sharp)}(0,g,0,\mathbf{u}) = 0$ for all $\mathbf{u}\in\mathbb{Z}^{nd}$ implies $g(H)=0$ almost surely. Let us suppose $g(H)\neq 0$. Then, there exists at least one $\mathbf{u}\in\mathbb{Z}^{nd}$ such that $\mathbf{M}^{(n,\sharp)}(0,g,0,\mathbf{u}) > 0$, and by (2.18), $\mathcal{M}^{(n,\sharp)}(p,g,T,u) > 0$. This implies

$$\sup_{\mathbf{y}\in\mathbb{Z}^{nd}} \mathcal{M}^{(n,\sharp)}(p,g,T,\mathbf{y}) > 0. \tag{2.19}$$

Definition 2.6. *We say $H^{(n)}_{\boldsymbol{\omega}}$ exhibits strong dynamical localization* in the $\sharp$-distance *in an open interval I if for all $g\in C^{\infty}_{c,+}(I)$ we have*

$$\sup_{\mathbf{y}\in\mathbb{Z}^{nd}} \mathbb{E}\{\sup_{t\in\mathbb{R}} M^{(n,\sharp)}_{\omega}(p,g,t,\mathbf{y})\} < \infty \tag{2.20}$$

for all $p\geq 0$. We say $H^{(n)}_{\boldsymbol{\omega}}$ exhibits strong dynamical localization in the $\sharp$-distance at an energy $E\in\mathbb{R}$ if there exists an open interval I containing E such that $H^{(n)}_{\boldsymbol{\omega}}$ exhibits strong dynamical localization in the $\sharp$-distance in I.

We denote by $\Sigma^{(n,\sharp)}_{\mathrm{SI}}$ the corresponding region of strong insulation, that is, the set of energies for which $H^{(n)}_{\boldsymbol{\omega}}$ exhibits strong dynamical localization in the $\sharp$-distance .

Definition 2.7. *Let $\mathbf{\Lambda} = \mathbf{\Lambda}^{(n)}_{\sharp;L}(\mathbf{y})$ be the n-particle $\sharp$-box of center $\mathbf{y}\in\mathbb{R}^{nd}$ and side-length L.*

i) *Given $\theta>0$, $E\in\mathbb{R}$, we say that $\mathbf{\Lambda}$ is (θ,E)*-suitable *if $E\notin\sigma(H^{(n)}_{\mathbf{\Lambda}})$ and*

$$|G_{\mathbf{\Lambda}}(\mathbf{u},\mathbf{v};E)| \leq L^{-\theta} \quad \text{for all } \mathbf{u}\in\mathbf{\Lambda}_{\sharp;L/3}(\mathbf{y}) \quad \text{and} \quad \mathbf{v}\in\partial_-\mathbf{\Lambda}. \tag{2.21}$$

Otherwise, $\mathbf{\Lambda}$ is (θ,E)-nonsuitable.

ii) *Given $m>0$, $E\in\mathbb{R}$, we say that $\mathbf{\Lambda}$ is (m,E)*-regular *if $E\notin\sigma(H^{(n)}_{\mathbf{\Lambda}})$ and*

$$|G_{\mathbf{\Lambda}}(\mathbf{u},\mathbf{v};E)| \leq e^{-mL/2} \quad \text{for all } \mathbf{u}\in\mathbf{\Lambda}_{\sharp;L/3}(\mathbf{y}) \text{ and } \mathbf{v}\in\partial_-\mathbf{\Lambda}. \tag{2.22}$$

Otherwise, $\mathbf{\Lambda}$ is (m,E)-nonregular.

iii) *Given $\zeta\in(0,1)$, $E\in\mathbb{R}$, we say that $\mathbf{\Lambda}$ is (ζ,E)-subexponentially suitable (SES) if, and only if, $E\notin\sigma\Big(H_{\mathbf{\Lambda}}\Big)$ and*

$$|G_{\mathbf{\Lambda}}(\mathbf{u},\mathbf{v};E)| \leq e^{-L^{\zeta}} \quad \text{for all } \mathbf{u}\in\mathbf{\Lambda}_{\sharp;L/3}(\mathbf{y}) \text{ and } \mathbf{v}\in\partial_-\mathbf{\Lambda}. \tag{2.23}$$

Otherwise, $\mathbf{\Lambda}$ is called (ζ,E)-nonsubexponentially suitable (nonSES).

Theorem 2.8. *Fix $\sharp\in\{\infty,S,H\}$, $n\in\mathbb{N}$, let $H^{(n)}_{\boldsymbol{\omega}}$ be the n-particle Anderson model and let $\mathcal{I}_n\subset\mathbb{R}$ be an open interval. Let $g_n\in C^{\infty}_{c,+}(\mathbb{R})$ such that $g_n\equiv 1$ on $\mathcal{I}_n$. If, for some $\alpha\geq 0$, $\theta>2nd$, and $p>p(\alpha,n,d,\theta)=(\theta+2nd)\alpha+6\theta+15nd$, we have*

$$\liminf_{T\to\infty}\frac{1}{T^{\alpha}}\sup_{\mathbf{y}\in\mathbb{Z}^{nd}}\mathcal{M}^{(n,\sharp)}\left(p,g_n,T,\mathbf{y}\right)<\infty, \tag{2.24}$$

then there exists a sequence of increasing length-scales L_k such that

$$\lim_{k\to\infty}\sup_{\mathbf{y}\in\mathbb{R}^{nd}}\mathbb{P}\{\mathbf{\Lambda}^{(n)}_{\sharp;L_k}(\mathbf{y}) \text{ is } (\theta,E)\text{-nonsuitable}\} = 0 \quad \text{for all } E\in\mathcal{I}_n. \tag{2.25}$$

We define the energy region of trivial transport $\Sigma_{\mathrm{TT}}^{(n,\sharp)}$ by

$$\Sigma_{\mathrm{TT}}^{(n,\sharp)} = \{E \in \mathbb{R},\, \beta^{-}_{(n,\sharp)}(E) = 0\}. \tag{2.26}$$

Note that we have

$$\mathbb{R} \setminus \Sigma^{(n)} \subset \Sigma_{\mathrm{SI}}^{(n,\sharp)} \subset \Sigma_{\mathrm{TT}}^{(n,\sharp)}. \tag{2.27}$$

If $n \geq 2$, the Multiscale Analysis for the n-particle Anderson model can only be performed in an interval at the bottom of the spectrum, and requires a Wegner estimate between boxes. This estimate is known for n-particles boxes separated in the Hausdorff distance (e.g., [KlN1, Corollary 2.4]), which allows the performance of the multiscale analysis for the n-particle Anderson model based on ∞-boxes, yielding dynamical localization in the Hausdorff distance (e.g., the discrete analogue of [KlN2, Theorem 1.6]).

The conclusions of Theorem 2.8 are statements about $\sharp$-boxes. [KlN2, Theorem 1.6] requires an initial step which is a statement about ∞-boxes, yielding dynamical localization in the Hausdorff distance. If we apply Theorem 2.8 to the conclusions of [KlN2, Theorem 1.6], we would obtain a statement about H-boxes. To go back we would need to perform a multiscale analysis using H-boxes.

There are technical problems with performing a multiscale analysis based on H-boxes. But these problems are not present for a multiscale analysis based on S-boxes. On the other hand, there is no Wegner estimate between boxes separated in the symmetrized distance, except for the case of two particles, where the symmetrized and the Hausdorff distance coincide. For this reason we will now restrict ourselves to the 2-particle Anderson model, for which we prove the following theorem, the analog of [KlN2, Theorem 1.6] for symmetrized two-particle boxes (see also [GK1]).

Theorem 2.9 (Bootstrap Multiscale Analysis for two-particle S-boxes)**.** *Let $H_{\boldsymbol{\omega}}^{(2)}$ be the 2-particle Anderson model. There exist $p_0(n) = p_0(n,d) > 0$, n=1,2, such that, given $\theta > 16d$ and energies $E^{(1)} > E^{(2)} > 0$, there exists $\mathcal{L} = \mathcal{L}(d, \|\rho\|_\infty, \theta, E^{(1)}, E^{(2)})$ such that if for some $L_0 \geq \mathcal{L}$ and $n = 1,2$ we have*

$$\sup_{x \in \mathbb{R}^{nd}} \mathbb{P}\left\{\mathbf{\Lambda}_{S;L_0}^{(n)}(\mathbf{x}) \text{ is } (\theta, E)\text{-nonsuitable}\right\} \leq p_0(n) \quad \text{for all } E \leq E^{(n)}, \tag{2.28}$$

then, given $0 < \zeta < 1$, we can find a length scale $L_\zeta = L_\zeta(d, \|\rho\|_\infty, \theta, E^{(1)}, E^{(2)}, L_0)$, $\delta_\zeta = \delta_\zeta(d, \|\rho\|_\infty, \theta, E^{(1)}, E^{(2)}, L_0) > 0$ and $m_\zeta = m_\zeta(L_\zeta, \delta_\zeta) > 0$, such that, for $n = 1,2$, we have that for every $E_1 < E^{(n)}$, $L \geq L_\zeta$ and all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{nd}$ with $d_S(\mathbf{a}, \mathbf{b}) > L$, we have

$$\mathbb{P}\left\{\mathbf{\Lambda}_{S;L}^{(n)}(\mathbf{a}) \text{ and } \mathbf{\Lambda}_{S;L}^{(n)}(\mathbf{b}) \text{ are } (m_\zeta, E)\text{-nonregular for some } E \in I(E_1)\right\} \leq e^{-L^\zeta}, \tag{2.29}$$

where $I(E_1) = [E_1 - \delta_\zeta, E_1 + \delta_\zeta] \cap (-\infty, E^{(n)}]$.

Unlike the one particle case, this multiscale analysis can only be performed in an interval at the bottom of the spectrum, and it requires also the performance of the multiscale analysis for one-particle in a slightly larger interval. For this reason $\Sigma_{\mathrm{MSA}}^{(2,S)}$, the set of energies where we can start the multiscale analysis for 2-particle S-boxes, has to be defined differently from [GK4, Definition 2.6]. We say $E^{(2)} \in \mathcal{E}_{\mathrm{MSA}}^{(2,S)}$ if $E^{(2)} > 0$ and there exists $E^{(1)} > E^{(2)}$ such that the condition (2.28) holds for

$n=1,2$ for some $L_0 \geq \mathcal{L}$, where $\mathcal{L}=\mathcal{L}(d,\|\rho\|_\infty,\theta,E^{(1)},E^{(2)})$ is as in Theorem 2.9. We set

$$\Sigma_{\mathrm{MSA}}^{(2,S)} = \left(-\infty, E_{\mathrm{MSA}}^{(2,S)}\right), \quad \text{where} \quad E_{\mathrm{MSA}}^{(2,S)} = \sup \mathcal{E}_{\mathrm{MSA}}^{(2,S)}. \tag{2.30}$$

For $n=1$, Theorem 2.8 corresponds to [GK4, Theorem 2.11]. Proceeding as in [KlN1, Theorem 3.21], we can extend the estimate (2.25), which holds for a sequence of length-scales L_k, to a corresponding estimate that holds for every large enough scale L (see Remark 4.13). This yields the following Corollary.

Corollary 2.10. *Assume the hypotheses of Theorem 2.8 hold for $n=1,2$ (with $\sharp = H = S$) on an interval $(-\infty,E_*)$, where $E_*>0$. Then $(-\infty,E_*)\subset \Sigma_{\mathrm{MSA}}^{(2,S)}$.*

We define:

$$\widetilde{\Sigma}_{\mathrm{TT}}^{(2,\sharp)} = (-\infty, E_{\mathrm{TT}}^{(2,\sharp)}) \quad \text{and} \quad \widetilde{\Sigma}_{\mathrm{SI}}^{(2,\sharp)} = (-\infty, E_{\mathrm{SI}}^{(2,\sharp)}), \tag{2.31}$$

where

$$\begin{aligned} E_{\mathrm{TT}}^{(2,\sharp)} &= \sup\left\{E; (-\infty,E)\subset \Sigma_{\mathrm{TT}}^{1}\cap \Sigma_{\mathrm{TT}}^{(2,\sharp)}\right\}, \\ E_{\mathrm{SI}}^{(2,\sharp)} &= \sup\left\{E; (-\infty,E)\subset \Sigma_{\mathrm{SI}}^{1}\cap \Sigma_{\mathrm{SI}}^{(2,\sharp)}\right\} \end{aligned} \tag{2.32}$$

Theorem 2.11. *Let $H_{\boldsymbol{\omega}}^{(2)}$ be the 2-particle Anderson model. Then, $\Sigma_{\mathrm{MSA}}^{(2,S)} \subset \widetilde{\Sigma}_{\mathrm{SI}}^{(2,S)}$. In particular, $\Sigma_{\mathrm{MSA}}^{(2,S)} \subset \widetilde{\Sigma}_{\mathrm{TT}}^{(2,S)}$.*

In the following Theorem, we show that for the two-particle Anderson model, the converse to Theorem 2.11 holds true. This gives a characterization of the metal-insulator transition for the two-particle Anderson model at the bottom of the spectrum, analogous to [GK4, Theorem 2.8].

Theorem 2.12. *Let $H_{\boldsymbol{\omega}}^{(2)}$ be the 2-particle Anderson model. Then, $\widetilde{\Sigma}_{\mathrm{TT}}^{(2,\sharp)} \subset \Sigma_{\mathrm{MSA}}^{(2,S)}$, which yields*

$$\Sigma_{\mathrm{MSA}}^{(2,S)} = \widetilde{\Sigma}_{\mathrm{SI}}^{(2,S)} = \widetilde{\Sigma}_{\mathrm{TT}}^{(2,\sharp)}. \tag{2.33}$$

Theorem 2.12 is a consequence of Theorems 2.8 and 2.11. It shows that for the 2-particle Anderson model, within the region of one-particle dynamical localization, slow transport (i.e., $\beta^-_{(2,S)}(p,E)$ small) at the bottom of the spectrum implies dynamical localization. In particular, it implies null transport (i.e., $\beta^-_{(2,S)}(p,E)=0$) in an interval at the bottom of the spectrum.

Remark 2.13. *We can obtain the following analogue of* [GK4, Theorem 2.10]*. Note that we have $E_{\mathrm{MSA}}^{(2,S)} \leq E_{\mathrm{MSA}}^{(1)}$ (recall (2.30)). As in the proof of* [GK4, Theorem 2.10]*, it follows from Theorem 2.8 applied to the one-particle setting that $\beta_1^-(E_{\mathrm{MSA}}^{(1)}) \geq \frac{1}{3d}$. If $E_{\mathrm{MSA}}^{(2,S)} = E_{\mathrm{MSA}}^{(1)}$ this is all we can say. If $E_{\mathrm{MSA}}^{(2,S)} < E_{\mathrm{MSA}}^{(1)}$, it follows from Theorem 2.8 that we must have $\beta^-_{(2,S)}(E_{\mathrm{MSA}}^{(2,S)}) \geq \frac{1}{20d}$.*

Remark 2.14. *Recently Chulaevsky obtained a Wegner estimate between boxes that are far apart in the symmetrized distance for all N, for a certain class of single site probability distributions* [ChS4, Theorem 2.2] *(see also* [Ch1]*). Under Chulaevsky's assumptions we would have Corollary 2.10 and Theorem 2.11, and therefore Theorem 2.12, for any $N\geq 2$.*

3. Proof of Theorem 2.8

We prove Theorem 2.8 following [GK4]. We need to adapt [GK4, Proposition 6.1 and Lemma 6.4] to the discrete setting and distance $\mathrm{dist}_\sharp$, $\sharp \in \{\infty, S, H\}$.

We start with [GK4, Proposition 6.1].

Theorem 3.1. *Let $H_{\boldsymbol{\omega}}^{(n)}$ be a random n-particle Schrödinger operator, $g \in C_{c,+}^{\infty}(\mathbb{R})$. For any $\mathbf{u} \in \mathbb{Z}^{nd}$, $p > 0$ and $T > 0$, we have*

$$\begin{aligned} &\mathcal{M}^{(n,\sharp)}(p, g, T, \mathbf{u}) \qquad (3.1)\\ &\quad = \frac{1}{\pi T} \int_{\mathbb{R}} \sum_{\mathbf{v} \in \mathbb{Z}^{nd}} \langle \mathrm{dist}_\sharp(\mathbf{v}, \mathbf{u}) \rangle^p \, \mathbb{E}\left(\left| \left\langle \delta_{\mathbf{v}}, G_{\boldsymbol{\omega}}\left(E + \tfrac{i}{T}\right) g\left(H_{\boldsymbol{\omega}}^{(n)}\right) \delta_{\mathbf{u}} \right\rangle \right|^2 \right) dE. \end{aligned}$$

The proof of Theorem 3.1 is similar to the proof of [GK4, Proposition 6.1], with the corresponding changes to the discrete setting, and considering the multiplication operator $\mathrm{dist}_\sharp(\mathbf{u}, \cdot)$ instead of the usual position operator $\langle \cdot \rangle = \mathrm{dist}_\infty(\mathbf{u}, \cdot)$, so we omit the proof.

The following is a generalization of [GK4, Lemma 6.4] to the discrete setting, we give a proof in Appendix C (see Lemma C.1) for the reader's convenience.

Lemma 3.2. *Let $H_{\boldsymbol{\omega}}^{(n)}$ be a random n-particle Schrödinger operator satisfying a Wegner estimate for $\sharp$-boxes of the form (A.1) in an open interval $\mathcal{I}$. Let us denote by $\boldsymbol{\Lambda}$ the n-particle $\sharp$-box $\boldsymbol{\Lambda}_{\sharp;L}^{(n)}$, where $\sharp \in \{\infty, S, H\}$. Let $p_0 > 0$ and $\gamma > nd$. There exists a scale $\mathcal{L} = \mathcal{L}(\gamma, n, d, \rho, p_0)$ such that, given $E \in \mathcal{I}$, $L \geq \mathcal{L}$, and subsets $B_1, B_2 \subset \boldsymbol{\Lambda}$ such that $\partial_- \boldsymbol{\Lambda} \subset B_2$, for each $a > 0$ and $\varepsilon > 0$ we have, for $\mathbf{u} \in B_1$ and $\mathbf{y} \in B_2$*

$$\begin{aligned} &\mathbb{P}\left(a < |G_{\boldsymbol{\Lambda}}(E + i\varepsilon; \mathbf{y}, \mathbf{u})|\right) \qquad (3.2)\\ &\qquad \leq \frac{4L^{\gamma+2nd}}{a} \sup_{\mathbf{k} \in \partial_+\boldsymbol{\Lambda} \cup B_2} \mathbb{E}\left(|G(E + i\varepsilon; \mathbf{k}, \mathbf{u})|\right) + p_0, \end{aligned}$$

and

$$\begin{aligned} &\mathbb{P}\left(a < |G_{\boldsymbol{\Lambda}}(E; \mathbf{y}, \mathbf{u})|\right) \qquad (3.3)\\ &\quad \leq \frac{8L^{\gamma+2nd}}{a} \sup_{\mathbf{k} \in \partial_+\boldsymbol{\Lambda} \cup B_2} \mathbb{E}\left(|G(E + i\varepsilon; \mathbf{k}, \mathbf{u})|\right) + 2^{3/2} C_n^{(\sharp)} \|\rho\|_\infty \sqrt{\frac{\varepsilon}{a}} L^{nd} + p_0. \end{aligned}$$

We are now ready to prove Theorem 2.8.

Proof of Theorem 2.8. Fix $n \geq 2$. We write for simplicity $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{\sharp;L}^{(n)}(\mathbf{x})$. We need to show that for $\theta > 0$ and any $\tilde{p}_0 := \tilde{p}_0(n) > 0$ there exists $\mathcal{L}_0$ such that, given $\mathcal{L}_1 \geq \mathcal{L}_0$, for some $L \geq \mathcal{L}_1$ we have

$$P_{E,L} := \mathbb{P}\left\{|G_{\boldsymbol{\Lambda}}(E; \mathbf{y}, \mathbf{u})| > L^{-\theta}\right\} < \tilde{p}_0, \qquad (3.4)$$

for all $\mathbf{u} \in \boldsymbol{\Lambda}_{\sharp;L/3}^{(n)}(\mathbf{x})$ and $\mathbf{y} \in \partial_- \boldsymbol{\Lambda}$ and all $E \in \mathcal{I}_n$, uniformly in $\mathbf{x} \in \mathbb{R}^{nd}$. We will proceed as in the proof of [GK4, Theorem 2.11]. Let $p_0 > 0$ and use Lemma 3.2 with $a = 8L^{-\theta}$, $B_1 = \boldsymbol{\Lambda}_{\sharp;L/3}$, $B_2 = \partial_-\boldsymbol{\Lambda}$. This gives the existence of $\mathcal{L}(\gamma, n, d, \rho, p_0)$ such that for $L \geq \mathcal{L}(\gamma, n, d, \rho, p_0)$, we have

$$P_{E,L} \leq L^{\gamma+2nd+\theta} \sup_{\mathbf{k} \in \partial_\pm \boldsymbol{\Lambda}} \mathbb{E}\left(|G(E + i\varepsilon; \mathbf{k}, \mathbf{u})|\right) + C_{\rho,n} \sqrt{\varepsilon} L^{nd+\theta/2} + p_0, \qquad (3.5)$$

where $C_{\rho,n} := C_n^{(\sharp)} \|\rho\|_\infty$ and $\partial_\pm \boldsymbol{\Lambda} := \partial_+ \boldsymbol{\Lambda} \cup \partial_- \boldsymbol{\Lambda}$. We will make the second term in the r.h.s. small by taking

$$L = L(\varepsilon) = \left(\frac{p_0}{2C_{\rho,n}\sqrt{\varepsilon}} \right)^{\frac{2}{\theta+2nd}}. \tag{3.6}$$

Then,

$$P_{E,L} \leq L^{\gamma+2nd+\theta} \sup_{\mathbf{k}\in\partial_\pm\boldsymbol{\Lambda}} \mathbb{E}\left(|G(E+i\varepsilon;\mathbf{k},\mathbf{u})|\right) + 2p_0. \tag{3.7}$$

We will split the first term in the r.h.s. of (3.7) as in [GK4, Eq. 6.31] using the fact that $g_n(H_\omega^{(n)}) \equiv 1$ on $\mathcal{I}_n$. We will drop the subscript from g_n for simplicity. Next, we use [GK4, Theorem A.5], which holds in the discrete setting, see [GK4, Remark 2.1], to obtain

$$L^{\gamma+2nd+\theta} \sup_{\mathbf{k}\in\partial_\pm\boldsymbol{\Lambda}} \mathbb{E}\left(\left|\left\langle \delta_{\mathbf{k}}, G(E+i\varepsilon)\left(1-g(H_\omega^{(n)})\right)\delta_{\mathbf{u}}\right\rangle\right|\right) < p_0, \tag{3.8}$$

where we used that $\operatorname{dist}_\infty(\mathbf{k},\mathbf{u}) > \operatorname{dist}_\sharp(\mathbf{k},\mathbf{u})$ and $\operatorname{dist}_\sharp(\mathbf{k},\mathbf{u}) > L/2$. We obtain

$$\begin{aligned} P_{E,L} &\leq L^{\gamma+2nd+\theta} \sup_{\mathbf{k}\in\partial_\pm\boldsymbol{\Lambda}} \mathbb{E}\left(\left|\left\langle \delta_{\mathbf{k}}, G(E+i\varepsilon)g(H_\omega^{(n)})\delta_{\mathbf{u}}\right\rangle\right|\right) + 3p_0 \\ &\leq C_p L^{-p/2+\gamma+2nd+\theta} \sup_{\mathbf{k}\in\partial_\pm\boldsymbol{\Lambda}} \mathbb{E}\left(\langle \operatorname{dist}_\sharp(\mathbf{k},\mathbf{u})\rangle^{p/2}\left|\left\langle \delta_{\mathbf{k}}, G(E+i\varepsilon)g(H_\omega^{(n)})\delta_{\mathbf{u}}\right\rangle\right|\right) \\ &\quad + 3p_0, \end{aligned} \tag{3.9}$$

where $C_p := 6^{p/2}$ and we used the fact that $\mathbf{k} \in \partial_\pm\boldsymbol{\Lambda}$ implies $\operatorname{dist}_\sharp(\mathbf{u},\mathbf{k}) > L/6$ for $L > 1$. We use Jensen's inequality to obtain

$$\begin{aligned} &\sup_{\mathbf{k}\in\partial_\pm\boldsymbol{\Lambda}} \mathbb{E}\left(\langle \operatorname{dist}_\sharp(\mathbf{k},\mathbf{u})\rangle^{p/2}\left|\left\langle \delta_{\mathbf{k}}, G(E+i\varepsilon)g(H_\omega^{(n)})\delta_{\mathbf{u}}\right\rangle\right|\right) \\ &\qquad \leq \mathbb{E}\left(\sum_{\mathbf{v}\in\mathbb{Z}^{nd}} \langle \operatorname{dist}_\sharp(\mathbf{v},\mathbf{u})\rangle^{p}\left|\left\langle \delta_{\mathbf{v}}, G(E+i\varepsilon)g(H_\omega^{(n)})\delta_{\mathbf{u}}\right\rangle\right|^2\right)^{1/2}. \end{aligned} \tag{3.10}$$

Next, we define

$$\begin{aligned} &A_{\mathbf{u},M,I,\varepsilon} := \\ &\left\{ E\in I : \mathbb{E}\left(\sum_{\mathbf{v}\in\mathbb{Z}^{nd}} \langle \operatorname{dist}_\sharp(\mathbf{v},\mathbf{u})\rangle^{p}\left|\left\langle \delta_{\mathbf{v}}, G(E+i\varepsilon)g(H_\omega^{(n)})\delta_{\mathbf{u}}\right\rangle\right|^2\right) \leq M\varepsilon^{-(\alpha+1)} \right\}. \end{aligned} \tag{3.11}$$

Taking $T = \varepsilon^{-1}$ and using Theorem 3.1, we get

$$|I \setminus A_{\mathbf{u},M,I,\varepsilon}| \leq \frac{\pi}{MT^\alpha} \sup_{\mathbf{u}\in\mathbb{Z}^{nd}} \mathcal{M}^{(n,\sharp)}(p,T,g,\mathbf{u}). \tag{3.12}$$

By our hypothesis (2.24), we can pick a sequence $T_j \to \infty$ such that, for j big enough, we have $\sup_{\mathbf{u}\in\mathbb{Z}^{nd}} \mathcal{M}^{(n,\sharp)}(p,g,T_j,\mathbf{u}) < CT_j^\alpha$ for some positive constant C. Then, for the corresponding sequence $\varepsilon_j := T_j^{-1} \to 0^+$ we have

$$|I \setminus A_{\mathbf{u},M,I,\varepsilon_j}| \leq \frac{C'}{M}, \tag{3.13}$$

where $C' = \pi C$. Note that this bound is uniform in $\mathbf{u}$.

For an $E \in I$ fixed, either $E \in A_{\mathbf{u},I,M,\varepsilon_j}$ or $E \in I \setminus A_{\mathbf{u},M,I,\varepsilon_j}$. In the first case we have,

$$P_{E,L_j} \leq C_p L_j^{-p/2+\gamma+2nd+\theta} M^{1/2} \varepsilon_j^{-(\alpha+1)/2} + 3p_0 \tag{3.14}$$

where we write $L_j := L(\varepsilon_j)$. If $E \in I \setminus A_{\mathbf{u},M,I,\varepsilon_j}$, by (3.13) there exists $E_{\mathbf{u}} \in A_{\mathbf{u},I,M,\varepsilon_j}$ such that $|E - E_{\mathbf{u}}| \leq \frac{C'}{M}$. Using the resolvent identity, we obtain

$$\mathbb{E}\left(\left|\langle \delta_{\mathbf{k}}, G(E+i\varepsilon_j) g(H_\omega^{(n)}) \delta_{\mathbf{u}} \rangle\right|\right) \leq \mathbb{E}\left(\left|\langle \delta_{\mathbf{k}}, G(E_{\mathbf{u}}+i\varepsilon_j) g(H_\omega^{(n)}) \delta_{\mathbf{u}} \rangle\right|\right) + \frac{C'}{M\varepsilon_j^2}. \tag{3.15}$$

It follows that

$$\begin{aligned} P_{E,L_j} \leq L_j^{\gamma+2nd+\theta} & \sup_{\mathbf{k} \in \partial_{\pm}\mathbf{\Lambda}} \mathbb{E}\left(\left|\left\langle \delta_{\mathbf{k}}, G(E_{\mathbf{u}}+i\varepsilon_j) g(H_\omega^{(n)}) \delta_{\mathbf{u}} \right\rangle\right|\right) \\ & + \frac{C' L_j^{\gamma+2nd+\theta}}{M\varepsilon_j^2} + 3p_0. \end{aligned} \tag{3.16}$$

We can bound the first term in the r.h.s. as in (3.9)-(3.14) and get

$$P_{E,L_j} \leq C_p L_j^{-p/2+\gamma+2nd+\theta} M^{1/2} \varepsilon_j^{-(\alpha+1)/2} + \frac{C' L_j^{\gamma+2nd+\theta}}{M\varepsilon_j^2} + 3p_0. \tag{3.17}$$

We set $M = L_j^{7\gamma+3\theta}$. Recalling (3.6) and $\gamma > nd$, we can take p such that $p > p(\alpha, n, d, \theta) = (\theta + 2nd)\alpha + 6\theta + 15nd$, and find γ and L_j larger than some scale $\mathcal{L} = \mathcal{L}(d, p, \alpha, \theta, \gamma, p_0, C_{\rho,n}, C')$, such that the r.h.s. of (3.17) is bounded by $5p_0$. Therefore, there exists a sequence $L_j \to \infty$ such that for L_j large enough and $p_0 < \tilde{p}_0/5$ we have $P_{E,L_j} < \tilde{p}_0$ uniformly on $\mathbf{x} \in \mathbb{R}^{nd}$. Since $\tilde{p}_0$ is arbitrary, we obtain the desired result. □

4. The Bootstrap Multiscale Analysis for symmetrized two-particle boxes

4.1. **Preliminaries.** Let $\mathbf{\Lambda}_{S;\boldsymbol{L}}^{(2)}(\mathbf{x})$ be the symmetrized two-particle rectangle with sides $\boldsymbol{L} = (L_1, L_2)$ and center $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^{2d}$ as in (2.9), and note that $\left|\mathbf{\Lambda}_{S;\boldsymbol{L}}^{(2)}(\mathbf{x})\right| \leq 2L^{2d}$, where $L = \max\{L_1, L_2\}$. We set

$$\begin{aligned} \Pi_j \mathbf{\Lambda}_{L_1,L_2}^{(2)}(\mathbf{x}) &= \Lambda_{L_j}(x_j) \quad \text{for} \quad j = 1, 2, \\ \Pi \mathbf{\Lambda}_{L_1,L_2}^{(2)}(\mathbf{x}) &= \Pi \mathbf{\Lambda}_{S;L_1,L_2}^{(2)}(\mathbf{x}) = \bigcup_{j=1,2} \Lambda_{L_j}(x_j). \end{aligned} \tag{4.1}$$

Definition 4.1. *A pair of symmetrized two-particle rectangles, $\mathbf{\Lambda}_{S;\boldsymbol{L}}^{(2)}(\mathbf{x})$ and $\mathbf{\Lambda}_{S;\boldsymbol{L}'}^{(2)}(\mathbf{y})$, are said to be fully separated if and only if*

$$\Pi \mathbf{\Lambda}_{S;\boldsymbol{L}}^{(2)}(\mathbf{x}) \bigcap \Pi \mathbf{\Lambda}_{S;\boldsymbol{L}'}^{(2)}(\mathbf{y}) = \emptyset. \tag{4.2}$$

The pair is said to be partially separated if and only if there exists $j \in \{1, 2\}$ such that

$$\textit{either } \Pi_j \mathbf{\Lambda}_{\boldsymbol{L}}^{(2)}(\mathbf{x}) \bigcap \Pi \mathbf{\Lambda}_{S;\boldsymbol{L}'}^{(2)}(\mathbf{y}) = \emptyset, \quad \textit{or } \Pi_j \mathbf{\Lambda}_{\boldsymbol{L}'}^{(2)}(\mathbf{y}) \bigcap \Pi \mathbf{\Lambda}_{S;\boldsymbol{L}}^{(2)}(\mathbf{x}) = \emptyset.$$

Note that events defined on a pair of fully separated symmetrized two-particle boxes are independent.

The following Wegner estimate for partially separated symmetrized two-particle rectangles can be proven in the same way as in [KlN1, Theorem 2.3 and Corollary 2.4], using Theorem A.1.

Proposition 4.2. *Consider a pair of partially separated symmetrized two-particle rectangles $\mathbf{\Lambda}^{(2)}_{S;\boldsymbol{L}}(\mathbf{x}) \subset \mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{x})$ and $\mathbf{\Lambda}^{(2)}_{S;\boldsymbol{L}'}(\mathbf{y}) \subset \mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{y})$. Then*

$$\mathbb{P}\Big\{d\left(\sigma(H_{\mathbf{\Lambda}^{(2)}_{S;\boldsymbol{L}}(\mathbf{x})}), \sigma(H_{\mathbf{\Lambda}^{(2)}_{S;\boldsymbol{L}'}(\mathbf{y})})\right) \le \varepsilon\Big\} \le 16\ \|\rho\|_\infty\ \varepsilon\, L^{4d}\ \text{ for all } \varepsilon > 0. \tag{4.3}$$

Let $\mathbf{\Lambda}_1 \subset \mathbf{\Lambda}_2 \subset \mathbb{Z}^{2d}$. We have $\partial^{\mathbf{\Lambda}_2}\mathbf{\Lambda}_1 \subseteq \partial\mathbf{\Lambda}_1$, so if $\mathbf{\Lambda}_1 = \mathbf{\Lambda}^{(2)}_{S;\ell}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^{2d}$, we have $\left|\partial^{\mathbf{\Lambda}_2}\mathbf{\Lambda}_1\right| \le |\partial\mathbf{\Lambda}_1| \le 2s_{2,d}\ell^{2d-1}$, for some constant $s_{2,d} > 0$.

Lemma 4.3. *Let $\mathbf{\Lambda} = \mathbf{\Lambda}^{(2)}_{S;\ell}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^{2d}$. If $(\mathbf{a}, \mathbf{b}) \in \partial\mathbf{\Lambda}$, we have*

$$\tfrac{\ell}{2} - 1 < \operatorname{dist}_S(\mathbf{a}, \mathbf{x}) \le \tfrac{\ell}{2} < \operatorname{dist}_S(\mathbf{b}, \mathbf{x}) \le \tfrac{\ell}{2} + 1. \tag{4.4}$$

Proof. We have $\mathbf{a} \in \mathbf{\Lambda}$, so $\operatorname{dist}_S(\mathbf{a}, \mathbf{x}) \le \frac{\ell}{2}$, $\mathbf{b} \notin \mathbf{\Lambda}$, so $\operatorname{dist}_S(\mathbf{b}, \mathbf{x}) > \frac{\ell}{2} \ge \operatorname{dist}_S(\mathbf{a}, \mathbf{x})$, and $\|\mathbf{a} - \mathbf{b}\|_1 = 1$, so $\|\mathbf{a} - \mathbf{b}\| = 1$.

Suppose $\operatorname{dist}_S(\mathbf{a}, \mathbf{x}) = \|\mathbf{a} - \mathbf{x}\| \le \|\mathbf{a} - \pi(\mathbf{x})\|$. Then

$$\|\mathbf{a} - \pi(\mathbf{x})\| \ge \|\mathbf{a} - \mathbf{x}\| \ge \|\mathbf{b} - \mathbf{x}\| - \|\mathbf{a} - \mathbf{b}\| > \frac{\ell}{2} - 1, \tag{4.5}$$

and we conclude that $\operatorname{dist}_S(\mathbf{a}, \mathbf{x}) > \frac{\ell}{2} - 1$. Moreover,

$$\operatorname{dist}_S(\mathbf{b}, \mathbf{x}) \le \|\mathbf{a} - \mathbf{x}\| + \|\mathbf{a} - \mathbf{b}\| \le \frac{\ell}{2} + 1. \tag{4.6}$$

$\square$

Let $\mathbf{\Lambda}_1 = \mathbf{\Lambda}^{(2)}_{S;\ell}(\mathbf{x})$ and $\mathbf{\Lambda}_2 = \mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{y})$ with $\mathbf{\Lambda}_1 \subset \mathbf{\Lambda}_2$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{2d}$. For $\mathbf{u} \in \mathbf{\Lambda}_1$, $\mathbf{v} \in \mathbf{\Lambda}_2 \setminus \mathbf{\Lambda}_1$, and $z \notin \sigma(H_{\mathbf{\Lambda}_1}) \cup \sigma(H_{\mathbf{\Lambda}_2})$, we have

$$(H_{\mathbf{\Lambda}_2} - z)^{-1}(\mathbf{u}, \mathbf{v}) = \sum_{(\mathbf{a},\mathbf{b}) \in \partial^{\mathbf{\Lambda}_2}(\mathbf{\Lambda}_1)} (H_{\mathbf{\Lambda}_1} - z)^{-1}(\mathbf{u}, \mathbf{a})\,(H_{\mathbf{\Lambda}_2} - z)^{-1}(\mathbf{b}, \mathbf{v})\,. \tag{4.7}$$

Hence, as a consequence of the geometric resolvent identity, we have

$$\begin{aligned} &\left|(H_{\mathbf{\Lambda}_2} - z)^{-1}(\mathbf{u}, \mathbf{v})\right| \\ &\quad\le \left|\partial^{\mathbf{\Lambda}_2}(\mathbf{\Lambda}_1)\right| \max_{(\mathbf{a},\mathbf{b}) \in \partial^{\mathbf{\Lambda}_2}(\mathbf{\Lambda}_1)} \left|(H_{\mathbf{\Lambda}_1} - z)^{-1}(\mathbf{u}, \mathbf{a})\,(H_{\mathbf{\Lambda}_2} - z)^{-1}(\mathbf{b}, \mathbf{v})\right| \\ &\quad\le 2s_{2,d}\ell^{2d-1} \max_{\mathbf{a} \in \partial_-^{\mathbf{\Lambda}_2}(\mathbf{\Lambda}_1)} \left|(H_{\mathbf{\Lambda}_1} - z)^{-1}(\mathbf{u}, \mathbf{a})\right| \left|(H_{\mathbf{\Lambda}_2} - z)^{-1}(\mathbf{b}_1, \mathbf{v})\right| \end{aligned} \tag{4.8}$$

for some $\mathbf{b}_1 \in \partial_+^{\mathbf{\Lambda}_2}(\mathbf{\Lambda}_1)$.

Definition 4.4. *Let $\mathbf{\Lambda}_S = \mathbf{\Lambda}^{(2)}_{S;\boldsymbol{L}}(\mathbf{x})$ with $\ell = \min(L_1, L_2)$, and $E \in \mathbb{R}$.*

(i) *Let $s > 0$. Then $\mathbf{\Lambda}_S$ is (E, s)-suitably resonant if and only if*

$$\operatorname{dist}\Big(\sigma(H_{\mathbf{\Lambda}_S}), E\Big) < \ell^{-s}. \tag{4.9}$$

Otherwise, $\mathbf{\Lambda}_S$ is (E, s)-suitably nonresonant.

(ii) *Let $\beta \in (0, 1)$. Then $\mathbf{\Lambda}_S$ is (E, β)-resonant if and only if*

$$\operatorname{dist}\Big(\sigma\big(H_{\mathbf{\Lambda}_S}\big), E\Big) < \tfrac{1}{2} e^{-\ell^{\beta}}. \tag{4.10}$$

Otherwise, $\mathbf{\Lambda}_S$ is (E, β)-nonresonant.

Let $\mathbf{\Lambda} = \mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{x})$ be a symmetrized two-particle box. We set

$$\Xi_{L,\ell}(\mathbf{x}) = \left\{\mathbf{x} + \left(\tfrac{\ell}{3} + 1\right) \mathbb{Z}^{2d}\right\} \cap \left\{\mathbf{y} \in \mathbb{R}^{2d};\ \|\mathbf{y} - \mathbf{x}\|_\infty \le \tfrac{L}{2} - \ell\right\}. \tag{4.11}$$

The symmetrized ℓ-suitable partial cover of $\mathbf{\Lambda}$ (or the ℓ-suitable partial cover of $\mathbf{\Lambda}$ for short) is the collection of symmetrized two-particle boxes

$$\mathcal{C}_{L,\ell}(\mathbf{x}) = \Big\{\mathbf{\Lambda}^{(2)}_{S;\ell}(\mathbf{y});\ \mathbf{y} \in \Xi_{L,\ell}(\mathbf{x})\Big\}. \tag{4.12}$$

Note that we have

$$\begin{gathered} \mathbf{\Lambda}^{(2)}_{S;\ell}(\mathbf{y}) \subset \mathbf{\Lambda} \quad \text{and} \quad \partial_- \mathbf{\Lambda} \cap \mathbf{\Lambda}^{(2)}_{S;\ell}(\mathbf{y}) = \emptyset \quad \text{for all} \quad \mathbf{y} \in \Xi_{L,\ell}(\mathbf{x}), \\ \# \mathcal{C}_{L,\ell}(\mathbf{x}) < \left(2\left(3\tfrac{L}{\ell} + 1\right)\right)^{2d}. \end{gathered} \tag{4.13}$$

Moreover, for every $\mathbf{u} \in \mathbf{\Lambda}^{(2)}_{S;L-2\ell}(\mathbf{x})$ there exists $\mathbf{y} \in \Xi_{L,\ell}(\mathbf{x})$ such that $\mathbf{u} \in \mathbf{\Lambda}^{(2)}_{S;\frac{\ell}{3}}(\mathbf{y})$.

Remark 4.5. *Elements in $\Xi_{L,\ell}(\mathbf{x})$ will be referred to as cells. Two cells, $\mathbf{a}, \mathbf{b} \in \Xi_{L,\ell}(\mathbf{x})$, are neighbors if and only if $\operatorname{dist}_\infty(\mathbf{a}, \mathbf{b}) = \frac{\ell}{3} + 1$. Moreover, by construction, if $\mathbf{y} \in \Xi_{L,\ell}(\mathbf{x})$ is sufficiently far away from the boundary of $\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{x})$, then for every $\mathbf{u} \in \partial_+ \left(\mathbf{\Lambda}^{(2)}_{S,\ell}(\mathbf{y})\right)$, we have $\mathbf{u} \in \mathbf{\Lambda}^{(2)}_{S,\ell}(\mathbf{a})$ for some $\mathbf{a} \in \Xi_{L,\ell}(\mathbf{x})$ that is a neighbor to $\mathbf{y}$.*

Definition 4.6. *Let $\mathbf{\Lambda} \subsetneq \mathbb{Z}^{2d}$. $\mathbf{\Lambda}$ is said to be non-interactive if and only if for every $\mathbf{y} = (y_1, y_2) \in \mathbf{\Lambda}$, we have*

$$\|y_1 - y_2\| > r_0 \tag{4.14}$$

Otherwise, it is said to be interactive .

Proposition 4.7. *Let $\mathbf{\Lambda}_1 = \Lambda_L(x_1) \times \Lambda_\ell(x_2)$ be a two-particle rectangle, $\mathbf{\Lambda} = \mathbf{\Lambda}_1 \cup \pi(\mathbf{\Lambda}_1)$ be a symmetrized two-particle rectangle with $L \ge \ell$. If $\|x_1 - x_2\| > L + r_0$, then we have the following:*

(i) *$\mathbf{\Lambda}$ is non-interactive,*
(ii) $\operatorname{dist}(\Lambda_L(x_1), \Lambda_\ell(x_2)) > r_0$,
(iii) $H_{\mathbf{\Lambda}_1} = H_{\Lambda_L(x_1)} \otimes I + I \otimes H_{\Lambda_\ell(x_2)}$,
(iv) $H_{\pi(\mathbf{\Lambda}_1)} = H_{\Lambda_\ell(x_2)} \otimes I + I \otimes H_{\Lambda_L(x_1)}$,
(v) $\sigma(H_{\mathbf{\Lambda}_1}) = \sigma\left(H_{\pi(\mathbf{\Lambda}_1)}\right) = \sigma\left(H_{\Lambda_L(x_1)}\right) + \sigma\left(H_{\Lambda_\ell(x_2)}\right)$,
(vi) $\operatorname{dist}(\mathbf{\Lambda}_1, \pi(\mathbf{\Lambda}_1)) > 1$
(vii) $H_{\mathbf{\Lambda}} = H_{\mathbf{\Lambda}_1} \oplus H_{\pi(\mathbf{\Lambda}_1)}$,
(viii) $\sigma(H_{\mathbf{\Lambda}}) = \sigma\left(H_{\Lambda_L(x_1)}\right) + \sigma\left(H_{\Lambda_\ell(x_2)}\right)$,
(ix) *for every $E \in \sigma(H_{\mathbf{\Lambda}})$,*

$$(H_{\mathbf{\Lambda}} - E)^{-1} = (H_{\mathbf{\Lambda}_1} - E)^{-1} \oplus \left(H_{\pi(\mathbf{\Lambda}_1)} - E\right)^{-1}. \tag{4.15}$$

Proof. It's clear that $(i) \Rightarrow (ii)$, $(vi) \Rightarrow (vii)$, and $(vii) \Rightarrow (viii)$ and (ix). By [KlN1, Lemma 2.7], $(ii) \Rightarrow (iii), (iv),$ and (v). $(ii) \Rightarrow (vi)$ since $r_0 \ge 1$. □

Remark 4.8. *Note that a two-particle box $\mathbf{\Lambda}_L^{(2)}(\mathbf{x})$ is non-interactive if and only if $\mathbf{\Lambda}_{S;L}^{(2)}(\mathbf{x})$ is non-interactive. Hence, if $\mathbf{\Lambda}_L^{(2)}(\mathbf{x})$ is non-interactive, then $\mathbf{\Lambda}_{S;L}^{(2)}(\mathbf{x})$ satisfies (ii)-(ix).*

We recall [KlN2, Lemma 3.9], which is an important ingredient for the two-particle bootstrap MSA (see also [KlN1, Lemma 2.8]).

Lemma 4.9. *Let $\mathbf{\Lambda}_L^{(2)}(\mathbf{x})$ be a two-particle box and $\mathbf{\Lambda}_{S;L}^{(2)}(\mathbf{x})$ be the corresponding symmetrized two-particle box. Let $E \leq E^{(2)}$ and $E^{(1)} > 0$ such that $E^{(1)} > E^{(2)}$. If $\|x_1 - x_2\| > L + r_0$ and L is sufficiently large,*

- (i) *Given $\theta > 2d+2$, suppose $\Lambda_L(x_1)$ is $(\theta,\, E-\mu)$-suitable for every $\mu \in \sigma\left(H_{\Lambda_L(x_2)}\right) \cap (-\infty, E^{(1)}]$ and $\Lambda_L(x_2)$ is $(\theta,\, E-\lambda)$-suitable for every $\lambda \in \sigma\left(H_{\Lambda_L(x_1)}\right) \cap (-\infty, E^{(1)}]$. Then $\mathbf{\Lambda}_{S;L}^{(2)}(\mathbf{x})$ is $\left(\frac{\theta}{2},\, E\right)$-suitable.*
- (ii) *Given $0 < m < \log\left(\frac{E^{(1)}-E^{(2)}}{4d}+1\right)$, suppose $\Lambda_L(x_1)$ is $(m,\, E-\mu)$-regular for every $\mu \in \sigma\left(H_{\Lambda_L(x_2)}\right) \cap (-\infty, E^{(1)}]$ and $\Lambda_L(x_2)$ is $(m,\, E-\lambda)$-regular for every $\lambda \in \sigma\left(H_{\Lambda_L(x_1)}\right) \cap (-\infty, E^{(1)}]$. Then $\mathbf{\Lambda}_{S;L}^{(2)}(\mathbf{x})$ is $\left(m - \frac{6(d+1)\log 2L}{L},\, E\right)$-regular.*
- (iii) *Given $0 < \zeta' < \zeta < 1$, suppose $\Lambda_L(x_1)$ is $(\zeta,\, E-\mu)$-SES for every $\mu \in \sigma\left(H_{\Lambda_L(x_2)}\right) \cap (-\infty, E^{(1)}]$ and $\Lambda_L(x_2)$ is $(\zeta,\, E-\lambda)$-SES for every $\lambda \in \sigma\left(H_{\Lambda_L(x_1)}\right) \cap (-\infty, E^{(1)}]$. Then $\mathbf{\Lambda}_{S;L}^{(2)}(\mathbf{x})$ is $(\zeta',\, E)$-SES.*

Proof. We prove (ii), the proofs of parts (i) and (iii) being similar. We begin by showing that $\mathbf{\Lambda}_L^{(2)}(\mathbf{x})$ is $\left(m - \frac{6(d+1)\log 2L}{L},\, E\right)$-regular.

Given $\mathbf{u} \in \mathbf{\Lambda}_{L/3}^{(2)}(\mathbf{x})$ and $\mathbf{y} \in \partial_- \mathbf{\Lambda}_L^{(2)}(\mathbf{x})$, $\|\mathbf{u}-\mathbf{y}\| > \frac{L}{6}$. Then either $\|u_1 - y_1\| > \frac{L}{6}$ or $\|u_2 - y_2\| > \frac{L}{6}$. Without loss of generality we assume the latter. We write $\sigma_1 := \sigma\left(H_{\Lambda_L(x_1)}\right) \cap (-\infty, E^{(1)}]$ and $\sigma_1^c := \sigma\left(H_{\Lambda_L(x_1)}\right) \cap (E^{(1)}, +\infty)$. We use the hypothesis, the Combes-Thomas estimate from Theorem B.1 (it is enough to take $\varepsilon = 1/2$ there) and the fact that $\operatorname{dist}\left(E-\mu, \sigma\left(H_{\Lambda_L(x_1)}\right)\right) > E^{(1)} - E^{(2)}$ for $\mu \in \sigma_1^c$ and $E \leq E^{(2)}$, to show that

$$
\begin{aligned}
|G_{\mathbf{\Lambda}}(E;\, \mathbf{u},\, \mathbf{y})| &\leq \sum_{\mu\in\sigma_1} \left|G_{\Lambda_L(x_2)}(E-\mu;\, u_2,\, y_2)\right| \\
&\qquad + \sum_{\mu\in\sigma_1^c} \left|G_{\Lambda_L(x_2)}(E-\mu;\, u_2,\, y_2)\right| \\
&\leq L^d e^{-m\|u_2-y_2\|} + \frac{2L^d}{E^{(1)}-E^{(2)}} e^{-\log\left(\frac{E^{(1)}-E^{(2)}}{4d}+1\right)\|u_2-y_2\|} \\
&\leq\ L^{d+1}\left(e^{-m\|u_2-y_2\|} + e^{-\log\left(\frac{E^{(1)}-E^{(2)}}{4d}+1\right)\|u_2-y_2\|}\right).
\end{aligned}
\tag{4.16}
$$

for L large enough. Taking $m < \log\left(\frac{E^{(1)}-E^{(2)}}{4d}+1\right)$ yields that, for L large enough, depending on $E^{(1)} - E^{(2)}$ and d, the box $\mathbf{\Lambda}_L^{(2)}(\mathbf{x})$ is $\left(m - \frac{6(d+1)\log 2L}{L},\, E\right)$-regular. Similarly, one can show the same holds for $\mathbf{\Lambda}_L^{(2)}(\pi(\mathbf{x}))$. By Equation (4.15), we conclude that $\mathbf{\Lambda}_{S;L}^{(2)}(\mathbf{x})$ is $\left(m - \frac{6(d+1)\log 2L}{L},\, E\right)$-regular. ∎

Proposition 4.10. *Given a pair of symmetrized two-particle boxes, $\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{x})$ and $\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{y})$ such that $d_S(\mathbf{x},\mathbf{y}) > L$, then the pair is partially separated.*

Proof. Since $\|\mathbf{x}-\mathbf{y}\| > L$ and $\|\pi(\mathbf{x})-\mathbf{y}\| > L$, we have

$$\begin{aligned} &\max\{\|x_1-y_1\|,\|x_2-y_2\|\} > L, \quad \text{and} \\ &\max\{\|x_1-y_2\|,\|x_2-y_1\|\} > L. \end{aligned} \tag{4.17}$$

There are four cases to consider here. If we take $\|x_1-y_1\| > L$ and $\|x_2-y_1\| > L$, then we have that the pair is partially separated since

$$\Pi_1\mathbf{\Lambda}^{(2)}_{L}(\mathbf{y})\bigcap\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{x}) = \emptyset. \tag{4.18}$$

The other three cases are handled in the same way. $\square$

Definition 4.11. *Given a pair of symmetrized two-particle boxes, $\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{x})$ and $\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{y})$. We say that the pair is L-distant if and only if*

$$\operatorname{dist}_S(\mathbf{x},\mathbf{y}) > 8L. \tag{4.19}$$

It follows from Proposition 4.10 that L-distant automatically implies partially separated.

Proposition 4.12. *Let $\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{x})$ and $\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{y})$ be a pair of symmetrized two-particle boxes. If the pair is interactive and L-distant, then $\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{x})$ and $\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{y})$ are fully separated, provided L is sufficiently large.*

Proof. Since $\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{x})$ and $\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{y})$ are interactive, thus there exists $\mathbf{a}\in\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{x})$ and $\mathbf{b}\in\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{y})$ such that

$$\|a_1-a_2\| \le r_0, \quad \text{and} \quad \|b_1-b_2\| \le r_0. \tag{4.20}$$

Thus, we have

$$\operatorname{dist}(\Lambda_L(x_1),\Lambda_L(x_2)) \le r_0 \quad \text{and} \quad \operatorname{dist}(\Lambda_L(y_1),\Lambda_L(y_2)) \le r_0. \tag{4.21}$$

Since $d_S(\mathbf{x},\mathbf{y}) > 8L$, we can proceed as in (4.17) and see that, in particular

$$\max\{\|x_1-y_1\|,\|x_2-y_2\|\} > 8L.$$

Thus, $\|x_1-y_1\| > 8L$ or $\|x_2-y_2\| > 8L$. Without loss of generality, let us assume $\|x_1-y_1\| > 8L$, then

$$\Lambda_{4L}(x_1)\bigcap\Lambda_{4L}(y_1) = \emptyset. \tag{4.22}$$

Moreover, (4.21) implies

$$\begin{aligned} &\Lambda_L(x_1)\bigcup\Lambda_L(x_2) \subseteq \Lambda_{3L+2r_0}(x_1) \subset \Lambda_{4L}(x_1) \ \text{ and} \\ &\qquad\Lambda_L(y_1)\bigcup\Lambda_L(y_2) \subseteq \Lambda_{3L+2r_0}(y_1) \subset \Lambda_{4L}(y_1), \end{aligned} \tag{4.23}$$

provided L is sufficiently large. Therefore,

$$\Pi\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{x})\bigcap\Pi\mathbf{\Lambda}^{(2)}_{S;L}(\mathbf{y}) = \emptyset. \tag{4.24}$$

$\square$

4.2. **The Bootstrap Multiscale Analysis.** The proof of Theorem 2.9 is similar to the proof of [KlN2, Theorem 1.6]. It uses the corresponding result for one-particle boxes from [GK1] and the definitions and results for symmetrized boxes introduced in the previous sections. We will state the steps needed for the proof and refer to [KlN1] and [KlN2] for details.

4.2.1. *The initial step for the MSA at the bottom of the spectrum.* That the hypotheses of Theorem 2.9 are satisfied at the bottom of the spectrum can be proven in the same way as [KlN2, Theorem 4.1] using Theorem B.1.

4.2.2. *Consequences of the one-particle case.* For the one-particle model, note that symmetrized boxes are usual boxes in the distance $\operatorname{dist}_\infty$. In this case we know from [GK1, Theorem 3.4] that there exists $E^{(1)} > 0$ such that, given $E^{(2)} < E^{(1)}$, for every $\tau \in (0,1)$ there is a length scale L_τ, $\delta_\tau > 0$, and $0 < m_\tau^* < \log\left(\frac{E^{(1)}-E^{(2)}}{4d} + 1\right)$, such that the following hold for all $E \le E^{(1)}$:

i) For all $L \ge L_\tau$ and $a \in \mathbb{R}^d$ we have
$$\mathbb{P}\Big\{\Lambda_L(a) \text{ is } (m_\tau^*,\, E)\text{-nonregular}\Big\} \le e^{-L^\tau}. \tag{4.25}$$

ii) Let $I(E) = [E-\delta_\tau, E+\delta_\tau] \subset (-\infty, E^{(1)}]$. For all $L \ge L_\tau$ and all pairs of disjoint one-particle boxes $\Lambda_L(a)$ and $\Lambda_L(b)$ we have
$$\begin{aligned}&\mathbb{P}\Big\{\exists\, E' \in I(E) \text{ so both } \Lambda_L(a) \text{ and } \Lambda_L(b) \text{ are } (m_\tau^*,\, E')\text{-nonregular}\Big\}\\ &\qquad \le e^{-L^\tau}.\end{aligned} \tag{4.26}$$

Remark 4.13. *The result of* [GK1] *holds for a sequence of scales. It holds for all sufficiently large scales using* [GK6, Lemma 3.16] *as in* [KlN1, Theorem 3.21].

This result yields probability estimates for non-interactive symmetrized two-particle boxes.

Lemma 4.14. *Let $\mathbf{\Lambda}_\ell^{(2)}(\mathbf{x}) = \Lambda_\ell(x_1) \times \Lambda_\ell(x_2)$ be a non-interactive two-particle box and $\tau \in (0,1)$. Then for ℓ large enough, depending on $E^{(1)}, E^{(2)}, d, \tau$, and for all $E \le E^{(2)}$ we have*
$$\begin{aligned}&\mathbb{P}\Big\{\mathbf{\Lambda}_{S;\ell}^{(2)}(\mathbf{x}) \textit{ is } (m_\tau^*(\ell), E)\textit{-nonregular}\Big\} \le \ell^{2d} e^{-\ell^\tau}, \textit{ with } m_\tau^*(\ell) = m_\tau^* - \tfrac{6(d+1)\log 2\ell}{\ell},\\ &\mathbb{P}\Big\{\mathbf{\Lambda}_{S;\ell}^{(2)}(\mathbf{x}) \textit{ is } (\theta,\, E)\textit{-nonsuitable}\Big\} \le \ell^{2d} e^{-\ell^\tau} \textit{for } \theta < \tfrac{\ell}{\log \ell}\tfrac{m_\tau^*(\ell)}{2},\\ &\mathbb{P}\Big\{\mathbf{\Lambda}_{S;\ell}^{(2)}(\mathbf{x}) \textit{ is } (\tau,\, E)\textit{-nonSES}\Big\} \le \ell^{2d} e^{-\ell^\tau}.\end{aligned} \tag{4.27}$$

Proof. Since both $\mathbf{\Lambda}_\ell^{(2)}(\mathbf{x})$ and $\mathbf{\Lambda}_\ell^{(2)}(\pi(\mathbf{x}))$ are non-interactive, the proof is a direct consequence of [KlN2, Lemma 5.1] adapted to the discrete setting and Lemma 4.9. $\square$

In what follows, we fix $\zeta, \tau, \beta, \zeta_0, \zeta_1, \zeta_2, \gamma$ such that
$$\begin{gathered}0 < \zeta < \tau < 1, \qquad \gamma > 1,\\ \zeta < \zeta_2 < \gamma\zeta_2 < \zeta_1 < \gamma\zeta_1 < \beta < \zeta_0 < \tau \quad \text{with} \quad \zeta\gamma^2 < \zeta_2.\end{gathered} \tag{4.28}$$

To aliviate the notation, in what follows we will omit the upperscript (2) from the notation of two-particle boxes and write simply $\mathbf{\Lambda}_L(\mathbf{x}), \mathbf{\Lambda}_{S;L}(\mathbf{x})$.

4.2.3. *The first Multiscale Analysis for symmetrized two-particle boxes.*

Proposition 4.15. *Let $E \leq E^{(2)}$, $\theta > 16d$, $0 < p < \theta - 8d + 2$, $Y \geq 66$, and $0 < p_0 < (6Y+2)^{-4d}$. If for some sufficiently large $L_0 \in \mathbb{N}$ we have*

$$\sup_{\mathbf{x}\in\mathbb{R}^{2d}} \mathbb{P}\Big\{\mathbf{\Lambda}_{S;L_0}(\mathbf{x}) \text{ is } (\theta,\, E)\text{-nonsuitable}\Big\} \leq p_0, \tag{4.29}$$

then, setting $L_{k+1} = Y\, L_k$, for $k = 0, 1, 2, ...$, there exists $K_0 \in \mathbb{N}$ such that for every $k \geq K_0$ we have

$$\sup_{\mathbf{x}\in\mathbb{R}^{2d}} \mathbb{P}\Big\{\mathbf{\Lambda}_{S;L_k}(\mathbf{x}) \text{ is } (\theta,\, E)\text{-nonsuitable}\Big\} \leq L_k^{-p}. \tag{4.30}$$

The proof relies on the following lemma, in the same way [KlN1, Proposition 3.2] relies on [KlN1, Lemma 3.3].

Lemma 4.16. *Let $E \in \mathbb{R}$, $s > 0$, $\theta > 4d - 2 + s$, $J \in \mathbb{N}$, $Y \geq 10 + 56J$, and $L = Y\ell$, and consider a symmetrized two-particle box $\mathbf{\Lambda} := \mathbf{\Lambda}_{S;L}(\mathbf{x})$ with the usual ℓ-suitable partial cover. Suppose*

i) $\mathbf{\Lambda}$ is (E,s)-suitably nonresonant.
ii) There exist at most J pairwise ℓ-distant symmetrized two-particle boxes in the suitable partial cover $\mathcal{C}_{L,\ell}(\mathbf{x})$ that are (θ, E)-nonsuitable.
iii) Every symmetrized two-particle box with center belonging to $\Xi_{L,\ell}(\mathbf{x})$ of side length $j\,(8\ell+1)$ with $j \in \{1,\dots,J\}$ is (E,s)-suitably nonresonant.

Then $\mathbf{\Lambda}$ is (θ, E)-suitable for L sufficiently large.

Proof. Without loss of generality, let us assume $\mathbf{\Lambda}_{S;\ell}(\mathbf{a}_1), \dots, \mathbf{\Lambda}_{S;\ell}(\mathbf{a}_J)$ are the J pairwise ℓ-distant symmetrized two-particle boxes in $\mathbf{\Lambda}$ that are (θ, E)-nonsuitable. This implies that if $\mathbf{\Lambda}_{S;\ell}(\mathbf{b})$ is ℓ-distant from $\mathbf{\Lambda}_{S;\ell}(\mathbf{a}_j)$ for every $j \in \{1,\dots,J\}$, and $\mathbf{\Lambda}_{S;\ell}(\mathbf{b}) \in \mathcal{C}_{L,\ell}(\mathbf{x})$, then $\mathbf{\Lambda}_{S;\ell}(\mathbf{b})$ must be (θ, E)-suitable.

Let us denote $\mathcal{T} = \{\mathbf{a}_1, \dots, \mathbf{a}_J\}$, and

$$\mathbf{\Lambda}_0 = \bigcup_{i=1,\dots,J} \mathbf{\Lambda}_{S;8\ell}(\mathbf{a}_i). \tag{4.31}$$

We can separate the set $\mathbf{\Lambda}_0$ into clusters $P_1, \dots, P_r$, which will be referred to as bad clusters, so that

i) for $i = 1, \dots, r$, $P_i \subset \mathbf{\Lambda}$, and each P_i is a symmetrized two-particle box of length $t_i(8\ell+1) \leq 9t_i\ell$ provided t_i is the maximum number of elements in $\mathcal{T}$ that belong to P_i,
ii) $\operatorname{dist}(P_i, P_j) > 1$ for $i \neq j$,
iii) $\bigcup_{i=1,\dots,J} \mathbf{\Lambda}_{S;\ell}(\mathbf{a}_i) \subset \bigcup_{i=1,\dots,r} P_r$,
iv) $\sum_{i=1,\dots,r} 9t_i\ell \leq 9J\ell$,
v) if $\mathbf{b} \notin P_i$ for every $i = 1, \dots, r$, and $\operatorname{dist}_S(\mathbf{x}, \mathbf{b}) \leq \frac{L}{2} - \ell$, then there exists $\mathbf{u} \in \Xi_{L,\ell}(\mathbf{x})$ such that $\mathbf{b} \in \mathbf{\Lambda}_{S;\ell}(\mathbf{u})$ and $\mathbf{\Lambda}_{S;\ell}(\mathbf{u})$ is (θ, E)-suitable.

Let us now fix $\mathbf{y} \in \partial_-\mathbf{\Lambda}$. Given $\mathbf{u} \in \mathbf{\Lambda}_{S,\frac{L}{3}}(\mathbf{x})$ with $\mathbf{u} \in \mathbf{\Lambda}_{S;\frac{\ell}{3}}(\mathbf{a})$ for some $\mathbf{a} \in \Xi_{L,\ell}(\mathbf{x})$, (note that $\mathbf{y} \notin \mathbf{\Lambda}_{S;\ell}(\mathbf{a})$ by construction), we have the following cases to consider:

a) if $\mathbf{a} \notin P_i$ for every $i \in \{1, \dots, r\}$, then $\mathbf{\Lambda}_{S;\ell}(\mathbf{a})$ is (θ, E)-suitable. Setting $\mathbf{\Lambda}_1 = \mathbf{\Lambda}_{S;\ell}(\mathbf{a})$ and using the geometric resolvent identity (4.8), we obtain

$$\begin{aligned}
&\left|(H_{\mathbf{\Lambda}} - E)^{-1}(\mathbf{u}, \mathbf{y})\right| \\
&\quad \leq 2s_{2,d}\ell^{2d-1} \max_{\mathbf{b} \in \partial_{-}^{\mathbf{\Lambda}}(\mathbf{\Lambda}_1)} \left|(H_{\mathbf{\Lambda}_1} - E)^{-1}(\mathbf{u}, \mathbf{b})\right| \left|(H_{\mathbf{\Lambda}} - E)^{-1}(\mathbf{b}_1, \mathbf{y})\right| \\
&\quad \leq 2s_{2,d}\ell^{2d-1}\ell^{-\theta} \left|(H_{\mathbf{\Lambda}} - E)^{-1}(\mathbf{b}_1, \mathbf{y})\right| \qquad (4.32)
\end{aligned}$$

with $\mathbf{b}_1 \in \partial_{+}^{\mathbf{\Lambda}}(\mathbf{\Lambda}_1)$. Then by construction, $\mathbf{b}_1 \in \mathbf{\Lambda}_{S;\ell}(\mathbf{v})$, where $\mathbf{v} \in \Xi_{L,\ell}(\mathbf{x})$ is a neighboring cell of $\mathbf{a}$.

b) if $\mathbf{a} \in P_i$ for some $i \in \{1, \dots, r\}$, and $\mathbf{y} \notin P_i$, then, using the fact that P_i is (E, s)-suitably nonresonant, we have

$$\begin{aligned}
&\left|(H_{\mathbf{\Lambda}} - E)^{-1}(\mathbf{u}, \mathbf{y})\right| \\
&\quad \leq 2s_{2,d}\ell^{2d-1} \max_{\mathbf{v} \in \partial_{-}^{\mathbf{\Lambda}}(P_i)} \left|(H_{P_i} - E)^{-1}(\mathbf{u}, \mathbf{v})\right| \left|(H_{\mathbf{\Lambda}} - E)^{-1}(\mathbf{v}_1, \mathbf{y})\right| \\
&\quad \leq 2s_{2,d}\ell^{2d-1}(Y\,\ell)^s \left|(H_{\mathbf{\Lambda}} - E)^{-1}(\mathbf{v}_1, \mathbf{y})\right| \qquad (4.33)
\end{aligned}$$

with $\mathbf{v}_1 \in \partial_{+}^{\mathbf{\Lambda}}(P_i)$. If there exists $\mathbf{b} \in \Xi_{L,\ell}(\mathbf{x})$ such that $\mathbf{v}_1 \in \mathbf{\Lambda}_{S;\frac{\ell}{3}}(\mathbf{b})$, $\mathbf{y} \notin \mathbf{\Lambda}_{S;\ell}(\mathbf{b})$, and $\mathbf{\Lambda}_{S;\ell}(\mathbf{b})$ is (θ, E)-suitable, then we can repeat Equation (4.32) with $\mathbf{v}_1$ replacing $\mathbf{u}$. Then

$$\begin{aligned}
&\left|(H_{\mathbf{\Lambda}} - E)^{-1}(\mathbf{u}, \mathbf{y})\right| \\
&\quad \leq 2s_{2,d}\ell^{2d-1}(Y\,\ell)^s \left|(H_{\mathbf{\Lambda}} - E)^{-1}(\mathbf{v}_1, \mathbf{y})\right| \\
&\quad\quad \leq 2s_{2,d}\ell^{2d-1}(Y\,\ell)^s 2s_{2,d}\ell^{2d-1}\ell^{-\theta} \left|(H_{\mathbf{\Lambda}} - E)^{-1}(\mathbf{v}_2, \mathbf{y})\right| \\
&\quad\quad\quad \leq \left|(H_{\mathbf{\Lambda}} - E)^{-1}(\mathbf{v}_2, \mathbf{y})\right|, \qquad (4.34)
\end{aligned}$$

where $\mathbf{v}_2 \in \partial_{+}^{\mathbf{\Lambda}}(\mathbf{\Lambda}_{S;\ell}(\mathbf{b}))$, provided $2s_{2,d}\ell^{2d-1}(Y\,\ell)^s 2s_{2,d}\ell^{2d-1}\ell^{-\theta} \leq 1$. Since by hypothesis, $\theta > 4d+s-2$, this can be achieved if ℓ is sufficiently large depending on $s_{2,d}, Y$ and s. Moreover, $\mathbf{v}_2$ belongs to a neighboring cell of $\mathbf{b}$. If such $\mathbf{b}$ does not exist, then we can always conclude

$$\left|(H_{\mathbf{\Lambda}} - E)^{-1}(\mathbf{u}, \mathbf{y})\right| \leq L^s. \qquad (4.35)$$

We will estimate $\left|(H_{\mathbf{\Lambda}} - E)^{-1}(\mathbf{u}, \mathbf{y})\right|$ with $\mathbf{u} \in \mathbf{\Lambda}_{S;\frac{L}{3}}(\mathbf{x})$. We first note that we can start with $\mathbf{u} \in \mathbf{\Lambda}_{S;\frac{L}{3}}(\mathbf{x})$ and apply either procedure (a) or (b) repeatedly. Since $\left(\frac{L}{2} - \ell - \frac{L}{6}\right)\left(\frac{\ell}{3} + 1\right)^{-1} = \left(\frac{L}{3} - \ell\right)\left(\frac{\ell}{3} + 1\right)^{-1} \geq \frac{L}{\ell+3} - 3$, for our purpose, the shortest path starting from $\mathbf{u}$ to the boundary of $\mathbf{\Lambda}$ has at least $\frac{L}{\ell+3} - 3$ cells. With $L = Y\ell$, thus we get

$$\tfrac{L}{\ell+3} - 3 = Y\left(\tfrac{\ell}{\ell+3}\right) - 3 > \tfrac{Y}{2} - 3. \qquad (4.36)$$

Then

$$\left|(H_{\mathbf{\Lambda}} - E)^{-1}(\mathbf{u}, \mathbf{y})\right| \leq \left(2s_{2,d}\ell^{2d-1}\ell^{-\theta}\right)^{N(Y)} L^s, \qquad (4.37)$$

where $N(Y)$ is the number of cells for which we are able to perform (a) without using the result for the control of a bad region. Having to account for the cells where we do not get anything due to the bad regions, which is, in the worst case, $9J\ell\left(\frac{\ell}{3}+1\right)^{-1}+J\leq 9J\ell\left(\frac{\ell}{3}\right)^{-1}+J=28J$, equation (4.36) gives us

$$N(Y)\geq \tfrac{Y}{2}-3-28J. \tag{4.38}$$

Our goal is to have $\left|(H_{\mathbf{\Lambda}}-E)^{-1}(\mathbf{x},\mathbf{y})\right|\leq L^{-\theta}$, so we would like

$$\left(2s_{2,d}\ell^{2d-1}\ell^{-\theta}\right)^{\frac{Y}{2}-3-28J}(Y\ell)^{s}\leq (Y\ell)^{-\theta}. \tag{4.39}$$

This can be achieved for ℓ sufficiently large if

$$(2d-1-\theta)\left(\tfrac{Y}{2}-3-28J\right)+s+\theta<0, \tag{4.40}$$

provided ℓ is sufficiently large depending on $s_{2,d},Y$ and s. Since $\theta>4d-2+s$, it follows that (4.40) is true if we have $\frac{Y}{2}-3-28J\geq 2$, that is, $Y\geq 10+56J$. ☐

Proof of Proposition 4.15. Using $0<p<\theta-8d+2$, we fix $s>0$ such that $4d+p<s<\theta-4d+2$.

Given a scale L, we set

$$p_L=\sup_{\mathbf{x}\in\mathbb{R}^{2d}}\mathbb{P}\Big\{\mathbf{\Lambda}_{S;L}(\mathbf{x})\text{ is }(\theta,\,E)\text{-nonsuitable}\Big\}. \tag{4.41}$$

Let $\mathbf{\Lambda}=\mathbf{\Lambda}_{S;L}(\mathbf{x})$ be a two-particle symmetrized box with an ℓ-suitable cover $\mathcal{C}_{L,\ell}(\mathbf{x})$, where $L=Y\ell$. We begin by defining several events:

$\mathcal{E}=\Big\{\mathbf{\Lambda}\ \text{ is }(\theta,E)\text{-nonsuitable}\Big\}$;

$\mathcal{A}$ is the event that there exists a non-interactive box, $\mathbf{\Lambda}_{S;\ell}(\mathbf{v})\in\mathcal{C}_{L,\ell}(\mathbf{x})$, that is (θ,E)-nonsuitable,

$\mathcal{W}_J$ is the event that $\mathbf{\Lambda}$ is (E,s)-suitably nonresonant and $\mathbf{\Lambda}_{S,j(8\ell+1)}(\mathbf{a})$ is (E,s)-suitably nonresonant for every $\mathbf{a}\in\Xi_{L,\ell}(\mathbf{x})$ and every $j\in\{1,\ldots,J\}$, and

$\mathcal{F}_J$ is the event that there are at most J pairwise ℓ-distant symmetrized two-particle boxes in $\mathcal{C}_{L,\ell}(\mathbf{x})$ that are (θ,E)-nonsuitable.

We take $Y\geq 10+56J$, with $J\in\mathbb{N}$ to be determined later. Then, by Proposition 4.16 we have

$$\mathbb{P}\left\{\mathcal{E}\right\}\leq\mathbb{P}\left\{\mathcal{W}_J^c\right\}+\mathbb{P}\left\{\mathcal{F}_J^c\right\}\leq\mathbb{P}\left\{\mathcal{W}_J^c\right\}+\mathbb{P}\left\{\mathcal{F}_J^c\cap\mathcal{A}^c\right\}+\mathbb{P}\left\{\mathcal{A}\right\}. \tag{4.42}$$

Note that

$$\mathcal{W}_J^c\subset\bigcup_{\substack{\mathbf{\Lambda}_{S;t}(\mathbf{y})=\mathbf{\Lambda}\text{ or }\mathbf{y}\in\Xi_{L,\ell}(\mathbf{x})\\ t\in\{j(8\ell+1)\,|\,j=1,\ldots,J\}}}\Big\{\operatorname{dist}\left(\sigma\left(H_{\mathbf{\Lambda}_{S;t}(\mathbf{y})}\right);E\right)\leq t^{-s}\Big\}. \tag{4.43}$$

With our choice of s, Theorem A.1 implies

$$\begin{aligned}\mathbb{P}\left\{\mathcal{W}_J^c\right\}&\leq\left(J(2\left(3\tfrac{L}{\ell}+1\right))^{2d}\right)\left(8\left\|\rho\right\|_\infty\ell^{-s}L^{2d}\right)\\&=J2^{2d}(3Y+1)^{2d}\left\|\rho\right\|_\infty Y^{4d}\ell^{4d-s}\leq\tfrac14(Y\ell)^{-p}=\tfrac14L^{-p},\end{aligned}$$

provided $4d+p<s$ and ℓ is sufficiently large. On the other hand, Lemma 4.14 yields, for ℓ is sufficiently large depending on $Y,d,p,\tau,\,E^{(1)},E^{(2)}$,

$$\mathbb{P}\left\{\mathcal{A}\right\}\leq(Y\ell)^{2d}\ell^{2d}e^{-\ell^{\tau}}\leq\tfrac14L^{-p}.$$

Hence

$$\mathbb{P}\Big\{\mathbf{\Lambda}_{S;L}(\mathbf{x})\ \text{ is }(\theta,E)\text{-nonsuitable}\Big\}\leq\tfrac12L^{-p}+\mathbb{P}\left\{\mathcal{F}_J^c\cap\mathcal{A}^c\right\}. \tag{4.44}$$

To bound $\mathbb{P}\left\{\mathcal{F}_J^c \cap \mathcal{A}^c\right\}$, we note that $\boldsymbol{\omega} \in \mathcal{F}_J^c \cap \mathcal{A}^c$ implies there are $J+1$ interactive pairwise ℓ-distant boxes in $\mathcal{C}_{L,\ell}(\mathbf{x})$ that are (θ, E)-nonsuitable. Hence, using Proposition 4.12, and recalling (4.12)-(4.13), we get

$$\mathbb{P}\left\{\mathcal{F}_J^c \cap \mathcal{A}^c\right\} \leq \tfrac{1}{2}\left(p_\ell\right)^{J+1}(6Y+2)^{2d(J+1)} = \tfrac{1}{2}\ell^{-p(J+1)}(6Y+2)^{2d(J+1)}. \tag{4.45}$$

Our goal is to have $\mathbb{P}\left\{\mathcal{F}_J^c \cap \mathcal{A}^c\right\} \leq \frac{1}{2}L^{-p}$, which means we just need to require

$$\tfrac{1}{2}\left(\ell^{-p}(6Y+2)^{2d}\right)^{J+1} \leq \tfrac{1}{2}(Y\ell)^{-p}. \tag{4.46}$$

This can be achieved if $J \geq 1$, provided ℓ is sufficiently large.

Next, our goal is to show that for sufficiently large L_0, letting $L_k = YL_{k-1}$, then there must exists a K_0 such that

$$p_{K_0} \leq L_{K_0}^{-p}. \tag{4.47}$$

We will proceed by seeing what happens when $K_0 \neq 0, 1, 2, \ldots$. From equation (4.44), taking ℓ to be large enough, then

$$\sup_{\mathbf{x}\in\mathbb{R}^{2d}} \mathbb{P}\Big\{\boldsymbol{\Lambda}_{S;L} \text{ is } (\theta,\, E)\text{-nonsuitable}\Big\} \leq \tfrac{1}{2}L^{-p} + \tfrac{1}{2}(6Y+2)^{2d(J+1)}p_\ell^{J+1}.$$

That is to say that for large enough ℓ,

$$2\sup_{\mathbf{x}\in\mathbb{R}^{2d}} \mathbb{P}\Big\{\boldsymbol{\Lambda}_{S;L}(\mathbf{x}) \text{ is } (\theta,\, E)\text{-nonsuitable}\Big\} \leq L^{-p} + \left((6Y+2)^{2d}p_\ell\right)^{J+1}. \tag{4.48}$$

This implies that for sufficiently large enough L_0, setting $p_k = p_{L_k}$, we have that for every $k \in \mathbb{N}$,

$$2p_{k+1} \leq (L_{k+1})^{-p} + \left((6Y+2)^{2d}p_k\right)^{J+1}. \tag{4.49}$$

If $K_0 = 0$, then we are done. If not, then we have that $(L_0)^{-p} < p_0$ and proceed to checking whether $K_0 = 1$. If $K_0 = 1$, then we are done. If not, then we must have that $(L_1)^{-p} < p_1$, and by equation (4.49) we must have $p_1 < \left((6Y+2)^{2d}p_0\right)^{J+1}$.

We now proceed to check whether $K_0 = 2$. If $K_0 = 2$, then we are done. If not, then we must have that $(L_2)^{-p} < p_2$, and by equation (4.49) we must have

$$\begin{aligned} p_2 < \left((6Y+2)^{2d}p_1\right)^{J+1} &< \left((6Y+2)^{2d}\left((6Y+2)^{2d}p_0\right)^{J+1}\right)^{J+1} \\ &= \left((6Y+2)^{(2d+2d(J+1))(J+1)}\right)p_0^{(J+1)^2} \end{aligned} \tag{4.50}$$

Thus $(L_2)^{-p} < \left((6Y+2)^{2d}\right)^{(J+1)^2+(J+1)} p_0^{(J+1)^2}$.

We will carry this process out until we can find such a K_0. We will proceed by contradiction and assume such a K_0 does not exists. It is clear that under such assumption, we get that for every $k \in \mathbb{N}$,

$$\begin{aligned} L_k^{-p} &< \left((6Y+2)^{2d}\right)^{(J+1)+(J+1)^2+\ldots+(J+1)^k} p_0^{(J+1)^k} \\ &= \left((6Y+2)^{2d}\right)^{(J+1)\left(1+(J+1)+\cdots+(J+1)^{k-1}\right)} p_0^{(J+1)^k} \\ &= \left((6Y+2)^{2d(J+1)}\right)^{\frac{(J+1)^k-1}{(J+1)-1}} p_0^{(J+1)^k}. \end{aligned} \tag{4.51}$$

Up to now, we only need $J \geq 1$ due to equation (4.46). Thus let us take $J = 1$ and by rewriting equation (4.51), we get $L_k^{-p} < \left((6Y+2)^{4d}\right)^{2^k-1} p_0^{2^k}$ for every $k \in \mathbb{N}$, i.e.

$$\left((6Y+2)^{4d}\right) L_0^{-p}(Y^{-p})^k < \left((6Y+2)^{4d} p_0\right)^{2^k} \quad \text{for every } k \in \mathbb{N}. \tag{4.52}$$

However, this cannot be true since $(6Y+2)^{4d} p_0 < 1$, and we have reached a contradiction.

□

4.2.4. *The second Multiscale Analysis for symmetrized two-particle boxes.*

Proposition 4.17. *Let $E \leq E^{(2)}$, $p > 0$, $0 < m_0 < m_\tau^*$, $1 < \gamma < 1 + \frac{p}{p+4d}$. If for some L_0 sufficiently large we have*

$$\sup_{\mathbf{x}\in\mathbb{R}^{2d}} \mathbb{P}\Big\{\mathbf{\Lambda}_{S;L_0}(\mathbf{x}) \text{ is } (m_0,\, E)\text{-nonregular}\Big\} \leq L_0^{-p}, \tag{4.53}$$

then, setting $L_{k+1} = L_k^\gamma$ for $k = 0, 1, \ldots$, we get

$$\sup_{\mathbf{x}\in\mathbb{R}^{2d}} \mathbb{P}\Big\{\mathbf{\Lambda}_{S;L_k}(\mathbf{x}) \text{ is } (\tfrac{m_0}{2},\, E)\text{-nonregular}\Big\} \leq L_k^{-p}, \tag{4.54}$$

for all $k = 0, 1, \ldots$.

The proof relies on the following lemma, in the same way [KlN1, Proposition 3.4] relies on [KlN1, Lemma 3.5].

Lemma 4.18. *Let $E \in \mathbb{R}$, $L = \ell^\gamma$, $p > 0$, $1 < \gamma < \frac{2p+4d}{p+4d}$, $0 < m_0 < m_\tau^*$, $J \in \mathbb{N}$, and*

$$m_\ell \in [\ell^{-\kappa}, m_0] \quad \text{where} \quad 0 < \kappa < \min\{\gamma - 1, \gamma(1-\beta), 1\} \tag{4.55}$$

Given a symmetrized two-particle box, $\mathbf{\Lambda} := \mathbf{\Lambda}_{S;L}(\mathbf{x})$ with the usual ℓ-suitable partial cover, suppose

i) $\mathbf{\Lambda}$ is (E,β)-nonresonant.

ii) There exist at most J pairwise ℓ-distant symmetrized two-particle box in the suitable partial cover $\mathcal{C}_{L,\ell}(\mathbf{x})$ that are (m_ℓ, E)-nonregular.

iii) Every symmetrized two-particle box with center belonging to $\Xi_{L,\ell}(\mathbf{x})$ whose length is $j\,(8\ell+1)$ with $j \in \{1,\ldots,J\}$ is (E,β)-nonresonant.

Then, for L large enough, $\mathbf{\Lambda}$ is (m_L, E)-regular, with

$$m_\ell > m_L \geq m_\ell - \tfrac{1}{2\ell^\kappa} \geq \tfrac{1}{L^\kappa}. \tag{4.56}$$

The proof of Lemma 4.18 is the same as Lemma 4.16, with the corresponding changes to *regular* boxes instead of *suitable* ones. The proof of Proposition 4.17 relies on Lemma 4.18 in the same way the proof [KlN1, Proposition 3.4] uses [KlN1, Lemma 3.5].

4.2.5. *The third Multiscale Analysis for symmetrized two-particle boxes.*

Proposition 4.19. *Given $0 < \zeta_1 < \zeta_0 < 1$, $Y = \max\left\{34^{\frac{1}{1-\zeta_0}}, 4^{\frac{1}{\zeta_0}}\right\}$, and $E \leq E^{(2)}$. If for some $L_0 \in \mathbb{N}$ sufficiently large we have*

$$\sup_{\mathbf{x}\in\mathbb{R}^{2d}} \mathbb{P}\Big\{\mathbf{\Lambda}_{S;L_0}(\mathbf{x}) \text{ is } (\zeta_0,\, E)\text{-nonSES}\Big\} \leq (6Y+2)^{-4d}, \tag{4.57}$$

then, setting $L_{k+1} = Y\, L_k$, for $k = 0, 1, 2, ...$, there exists $K_0 \in \mathbb{N}$ such that for every $k \geq K_0$ we have

$$\sup_{\mathbf{x}\in\mathbb{R}^{2d}} \mathbb{P}\Big\{\mathbf{\Lambda}_{S;L_k}(\mathbf{x}) \text{ is } (\zeta_0,\, E)\text{-nonSES}\Big\} \leq e^{-L_k^{\zeta_1}}. \tag{4.58}$$

The proof of Proposition 4.19 relies on the following lemma, in the same way as [KlN1, Proposition 3.6] is based on [KlN1, Lemma 3.7].

Lemma 4.20. *Let $0 < \zeta_1 < \zeta_0 < 1$, $Y = \max\left\{34^{\frac{1}{1-\zeta_0}}, 4^{\frac{1}{\zeta_0}}\right\}$, $L = Y\ell$, $J \in \mathbb{N}$, and $E \in \mathbb{R}$. Given a symmetrized two-particle box, $\mathbf{\Lambda} := \mathbf{\Lambda}_{S;L}(\mathbf{x})$, with the usual ℓ-suitable partial cover. Suppose*

i) $\mathbf{\Lambda}$ is (E, β)-nonresonant.
ii) There exists at most J pairwise ℓ-distant symmetrized two-particle box in the suitable partial cover $\mathcal{C}_{L,\ell}(\mathbf{x})$ that are (ζ_0, E)-nonSES.
iii) Every symmetrized two-particle box with center belonging to $\Xi_{L,\ell}(\mathbf{x})$ whose length is $j\,(8\ell + 1)$ with $j \in \{1, \ldots, J\}$ is (E, β)-nonresonant.

Then $\mathbf{\Lambda}$ is (ζ_0, E)-SES for L large enough.

Lemma 4.20 can be proved in the same way as Lemma 4.16 and Lemma 4.18 adapted to nonSES boxes. We refer the reader to [KlN1, Lemma 3.7] for details.

4.2.6. *The fourth Multiscale Analysis for symetrized two-particle boxes.* In this section we proceed with the energy-interval Multiscale Analysis. We fix $\zeta, \tau, \beta, \zeta_1, \zeta_2, \gamma$ as in (4.28) and take $L = \ell^\gamma$.

Definition 4.21. *Let $\mathbf{\Lambda}_{S;L}(\mathbf{x})$ be a non-interactive symmetrized two-particle box with the usual ℓ suitable partial cover, and consider an energy $E \leq E^{(2)}$. Then:*

(i) *$\mathbf{\Lambda}_{S;L}(\mathbf{x})$ is not E-Lregular (for left regular) if and only if there are two partially separated boxes in $\mathcal{C}_{L,\ell}(x_1)$ that are $(m^*_\tau,\, E - \mu)$-nonregular for some $\mu \in \sigma\left(H_{\Lambda_L(x_2)}\right) \cap (-\infty, E^{(1)}]$.*
(ii) *$\mathbf{\Lambda}_{S;L}(\mathbf{x})$ is not E-Rregular (for right regular) if and only if there are two partially separated boxes in $\mathcal{C}_{L,\ell}(x_2)$ that are $(m^*_\tau,\, E - \lambda)$-nonregular for some $\lambda \in \sigma\left(H_{\Lambda_L(x_1)}\right) \cap (-\infty, E^{(1)}]$.*
(iii) *$\mathbf{\Lambda}_{S;L}(\mathbf{x})$ is E-preregular if and only if $\mathbf{\Lambda}_{S;L}(\mathbf{x})$ is E-Lregular and E-Rregular.*

The following Lemma can be proven as in [KlN1, Lemma 3.15],

Lemma 4.22. *Let $E_0 \leq E^{(2)}$, $I = [E_0 - \delta_\tau,\, E_0 + \delta_\tau] \subset (-\infty, E^{(1)}]$, and consider a non-interactive symmetrized two-particle box $\mathbf{\Lambda}_{S;L}(\mathbf{x})$. Then*

(i) $\mathbb{P}\left\{\mathbf{\Lambda}_{S;L}(\mathbf{x}) \text{ is not } E\text{-Lregular for some } E \in I\right\} \leq L^{3d} e^{-\ell^\tau}$,
(ii) $\mathbb{P}\left\{\mathbf{\Lambda}_{S;L}(\mathbf{x}) \text{ is not } E\text{-Rregular for some } E \in I\right\} \leq L^{3d} e^{-\ell^\tau}$.

We conclude that

$$\mathbb{P}\Big\{\mathbf{\Lambda}_{S;L}(\mathbf{x}) \text{ is not } E\text{-preregular for some } E \in I\Big\} \leq 2L^{3d} e^{-\ell^\tau}. \tag{4.59}$$

Definition 4.23. *Let $\mathbf{\Lambda} = \mathbf{\Lambda}_{S,L}(\mathbf{x})$ be a non-interactive two-particle box, and consider an energy $E \leq E^{(2)}$. Then:*

(i) *$\mathbf{\Lambda}_{S,L}(\mathbf{x})$ is E-left nonresonant (or LNR) if and only if every box $\Lambda_{9\ell}(a) \subseteq \Lambda_L(x_1)$, with $a \in \Xi_{L,\ell}(x_1)$, is $(E - \mu, \beta)$-nonresonant for every $\mu \in \sigma\left(H_{\Lambda_L(x_2)}\right) \cap (-\infty, E^{(1)}]$. Otherwise we say $\mathbf{\Lambda}_{S,L}(\mathbf{x})$ is E-left resonant (or LR).*

(ii) $\mathbf{\Lambda}_{S,L}(\mathbf{x})$ *is E-right nonresonant (or RNR) if and only if every box $\Lambda_{9\ell}(a) \subseteq \Lambda_L(x_2)$, with $a \in \Xi_{L,\ell}(x_2)$, is $(E-\lambda, \beta)$-nonresonant for every $\lambda \in \sigma\left(H_{\Lambda_L(x_1)}\right) \cap (-\infty, E^{(1)}]$. Otherwise we say $\mathbf{\Lambda}_{S,L}(\mathbf{x})$ is E-right resonant (or RR).*

(iii) *We say $\mathbf{\Lambda}_{S,L}(\mathbf{x})$ is E-highly nonresonant (or HNR) if and only if $\mathbf{\Lambda}$ is E-nonresonant, that is, E-LNR and E-RNR.*

Lemma 4.24. *Let $E \in \mathbb{R}$, and $\mathbf{\Lambda}_{S;L}(\mathbf{x})$ be a non-interactive two-particle box. Assume that the following are true:*

(i) $\mathbf{\Lambda}_{S;L}(\mathbf{x})$ *is E-HNR.*

(ii) $\mathbf{\Lambda}_{S;L}(\mathbf{x})$ *is E-preregular.*

Then $\mathbf{\Lambda}_{S;L}(\mathbf{x})$ is $(m(L), E)$-regular, where

$$m(L) = m_\tau^* - \frac{1}{2L^\kappa} - \frac{6(d+1)\log 2L}{L}, \tag{4.60}$$

where κ is defined in Lemma 4.18.

The proof of the statement for $\mathbf{\Lambda}_L(\mathbf{x})$ and $\mathbf{\Lambda}_L(\pi(\mathbf{x}))$ follows the arguments in [KlN2, Lemma 5.17]. The desired result for $\mathbf{\Lambda}_{S;L}(\mathbf{x})$ is a consequence of (4.15).

Lemma 4.25. *Let $E \le E^{(2)}$ and $\mathbf{\Lambda}_{S;L}(\mathbf{x})$ be a non-interactive symmetrized two-particle box.*

(i) *If $\mathbf{\Lambda}_{S;L}(\mathbf{x})$ is E-right resonant, then there exists a two-particle rectangle*

$$\mathbf{\Lambda}_1 = \Lambda_L(x_1) \times \Lambda_{9\ell}(u), \tag{4.61}$$

where $u \in \Xi_{L,\ell}(x_2)$, and $\mathbf{\Lambda}_{9\ell}(u) \subseteq \Lambda_L(x_2)$, such that

$$\operatorname{dist}\left(\sigma\left(H_{\mathbf{\Lambda}_1}\right), E\right) < \tfrac{1}{2} e^{-(9\ell)^\beta} \le \tfrac{1}{2} e^{-\ell^\beta}. \tag{4.62}$$

Therefore,

$$\operatorname{dist}\bigl(\sigma\left(H_{\mathbf{\Lambda}}\right), E\bigr) = \operatorname{dist}\bigl(\sigma\left(H_{\mathbf{\Lambda}_1}\right), E\bigr) < \tfrac{1}{2} e^{-(9\ell)^\beta} \le \tfrac{1}{2} e^{-\ell^\beta}. \tag{4.63}$$

(ii) *If $\mathbf{\Lambda}_{S;L}(\mathbf{x})$ is E-left resonant, then there exists a two-particle rectangle*

$$\mathbf{\Lambda}_1 = \Lambda_{9\ell}(u) \times \mathbf{\Lambda}_L(x_2), \tag{4.64}$$

where $u \in \Xi_{L,\ell}(x_1)$, and $\mathbf{\Lambda}_{9\ell}(u) \subseteq \mathbf{\Lambda}_L(x_1)$, such that

$$\operatorname{dist}\bigl(\sigma\left(H_{\mathbf{\Lambda}_1}\right), E\bigr) < \tfrac{1}{2} e^{-(9\ell)^\beta} \le \tfrac{1}{2} e^{-\ell^\beta}. \tag{4.65}$$

Therefore,

$$\operatorname{dist}\left(\sigma\left(H_{\mathbf{\Lambda}}\right), E\right) = \operatorname{dist}\left(\sigma\left(H_{\mathbf{\Lambda}_1}\right), E\right) < \tfrac{1}{2} e^{-(9\ell)^\beta} \le \tfrac{1}{2} e^{-\ell^\beta}. \tag{4.66}$$

The proof follows the arguments in the proof of [KlN2, Lemma 5.18], plus (4.15).

We now state the energy interval multiscale analysis. Given $m > 0$, $L \in \mathbb{N}$, $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^{2d}$, and an interval I, we define the event

$$R\left(m, I, \mathbf{x}, \mathbf{y}, L\right) = \tag{4.67}$$
$$\left\{\exists\, E \in I \text{ such that } \mathbf{\Lambda}_{S;L}(\mathbf{x}) \text{ and } \mathbf{\Lambda}_{S;L}(\mathbf{y}) \text{ are not } (m, E)\text{-regular}\right\}.$$

Proposition 4.26. *Let ζ, τ, β, ζ_1, ζ_2, γ as in (4.28). Given $E \le E^{(2)}$, there exists a length scale $\mathcal{Z}$ such that if for some $L_0 \ge \mathcal{Z}$ we can verify*

$$\sup_{\mathbf{x}\in\mathbb{R}^{2d}} \mathbb{P}\Bigl\{\mathbf{\Lambda}_{S;L_0}(\mathbf{x}) \textit{ is } (2L_0^{\zeta_0-1},\, E)\textit{-nonregular}\Bigr\} \le e^{-L_0^{\zeta_1}}, \tag{4.68}$$

with $m_0 := (2L_0^{\zeta_0 - 1} - 6\log 2L_0^{-1}) < m_\tau^*$, *then, there exists* $\delta = \delta(L_0, \zeta_0, \beta)$ *such that, setting* $I = [E-\delta, E+\delta] \subset (-\infty, E^{(2)}]$ *and* $L_{k+1} = L_k^\gamma = L_0^{\gamma^k}$ *for* $k = 0, 1, 2, ...,$ *we have*

$$\mathbb{P}\Big\{ R\left(\frac{m_0}{2},\, I,\, \mathbf{x},\, \mathbf{y},\, L_k\right)\Big\} \le e^{-L_k^{\zeta_2}} \tag{4.69}$$

for every pair of partially separated two-particle symmetrized boxes $\mathbf{\Lambda}_{S;L_k}(\mathbf{x})$ *and* $\mathbf{\Lambda}_{S;L_k}(\mathbf{y})$.

Proof. We can proceed as in [KlN1, Proposition 3.13] to deduce from the hypothesis that, setting

$$\delta = \frac{1}{2} e^{-L_0^{\zeta_0} - 2L_0^\beta},$$

we have

$$\sup_{x\in\mathbb{R}^{2d}} \mathbb{P}\Big\{\exists E \in I \text{ such that } \mathbf{\Lambda}_{S;L_0}(\mathbf{x}) \text{ is } (m_0, E)\text{-nonregular}\Big\} \le e^{-L_0^{\zeta_1}}. \tag{4.70}$$

Then, we can argue as in [GK1, Eq. 5.37 to 5.38] and obtain

$$\mathbb{P}\Big\{ R\left(m_0,\, I,\, \mathbf{x},\, \mathbf{y},\, L_0\right)\Big\} \le e^{-L_0^{\zeta_2}} \tag{4.71}$$

for every pair of partially separated two-particle symmetrized boxes $\mathbf{\Lambda}_{S;L_0}(\mathbf{x})$ and $\mathbf{\Lambda}_{S;L_0}(\mathbf{y})$.

Given ℓ (sufficiently large) and $0 < m_\ell < m_\tau$, we set $L = \ell^\gamma$ and take m_L as in (4.56). If ℓ is large, we have $m(\ell) > m_\ell$, where $m(\ell)$ is given in (4.60), and conclude that $m(L) \ge m(\ell) > m_\ell > m_L$.

We start by showing that if

$$\mathbb{P}\Big\{ R\left(m_\ell,\, I,\, \mathbf{x},\, \mathbf{y},\, \ell\right)\Big\} \le e^{-\ell^{\zeta_2}} \tag{4.72}$$

for every pair of partially separated two-particle boxes $\mathbf{\Lambda}_{S;\ell}(\mathbf{x})$ and $\mathbf{\Lambda}_{S;\ell}(\mathbf{y})$, then

$$\mathbb{P}\Big\{ R\left(m_L,\, I,\, \mathbf{x},\, \mathbf{y},\, L\right)\Big\} \le e^{-L^{\zeta_2}} \tag{4.73}$$

for every pair of partially separated symmetrized two-particle boxes $\mathbf{\Lambda}_{S;L}(\mathbf{x})$ and $\mathbf{\Lambda}_{S;L}(\mathbf{y})$.

Let $\mathbf{\Lambda}_{S;L}(\mathbf{x})$ and $\mathbf{\Lambda}_{S;L}(\mathbf{y})$ be a pair of partially separated symmetrized two-particle boxes. Let $J \in 2\mathbb{N}$. Let $\mathcal{B}_J$ be the the event that there exists $E \in I$ such that either $\mathcal{C}_{L,\ell}(\mathbf{x})$ or $\mathcal{C}_{L,\ell}(\mathbf{y})$ contains J pairwise ℓ-distant interactive symmetrized boxes that are (m_ℓ, E)-nonregular, and let $\mathcal{A}$ be the event that there exists $E \in I$ such that either $\mathcal{C}_{L,\ell}(\mathbf{x})$ or $\mathcal{C}_{L,\ell}(\mathbf{y})$ contains one non-interactive symmetrized box that is not E-preregular. If $\boldsymbol{\omega} \in \mathcal{B}_J^c \cap \mathcal{A}^c$, then for all $E \in I$ the following holds:

(i) $\mathcal{C}_{L,\ell}(\mathbf{x})$ and $\mathcal{C}_{L,\ell}(\mathbf{y})$ contain at most $J-1$ pairwise ℓ-distant interactive (m_ℓ, E)-nonregular boxes.

(ii) Every interactive box in $\mathcal{C}_{L,\ell}(\mathbf{x})$ and $\mathcal{C}_{L,\ell}(\mathbf{y})$ is E-preregular.

We also define the event

$$\mathcal{U}_J = \bigcup_{\mathbf{\Lambda}' \in \mathcal{M}_{\mathbf{x}}, \mathbf{\Lambda}'' \in \mathcal{M}_{\mathbf{y}}} \left\{ \operatorname{dist}\left(\sigma(H_{\mathbf{\Lambda}'}), \sigma(H_{\mathbf{\Lambda}''})\right) < e^{-\ell^\beta} \right\}, \tag{4.74}$$

where, given a symmetrized two-particle box $\mathbf{\Lambda}_{S,L}(\mathbf{a})$, by $\mathcal{M}_{\mathbf{a}}$ we denote the collection of all symmetrized two-particle rectangles of the following three types:

(i) $\mathbf{\Lambda}_{S,L}(\mathbf{a})$,

(ii) $\mathbf{\Lambda}_{S;9j\ell}(\mathbf{u}) \subseteq \mathbf{\Lambda}_{S,L}(\mathbf{a})$, where $\mathbf{u} \in \Xi_{L,\ell}(\mathbf{a})$, and $j \in \{1, 2, \ldots, J\}$,

(iii) $\mathbf{\Lambda}$ is a symmetrized two-particle rectangle generated by a two-particle rectangle either of the form $\Lambda_{9\ell}(v) \times \Lambda_L(a_2)$, where $v \in \Xi_{L,\ell}(a_1)$, or $\Lambda_L(a_1) \times \Lambda_{9\ell}(v)$, where $v \in \Xi_{L,\ell}(a_2)$.

It is clear that if $J \geq 3$, then $|\mathcal{M}_{\mathbf{a}}| < J\left(\frac{6L}{\ell}+2\right)^{2d} + 2\left(\frac{6L}{\ell}+2\right)^{d} + 1 < (J+1)\left(\frac{6L}{\ell}+2\right)^{2d}$, and hence it follows from Proposition 4.2 that

$$\mathbb{P}\left\{\mathcal{U}_{\mathcal{J}}\right\} \leq \left((J+1)\left(\tfrac{6L}{\ell}+2\right)^{2d}\right)^{2}\left(16\left\|\rho\right\|_{\infty} L^{4d} e^{-\ell^{\beta}}\right). \tag{4.75}$$

Note that for $\boldsymbol{\omega} \in \mathcal{U}_J^c$ and $E \in I$, either every symmetrized two-particle rectangle in $\mathcal{M}_{\mathbf{x}}$ is (E,β)-nonresonant or every symmetrized two-particle rectangle in $\mathcal{M}_{\mathbf{y}}$ is (E,β)-nonresonant.

Let $\boldsymbol{\omega} \in \mathcal{B}_J^c \cap \mathcal{A}^c \cap \mathcal{U}_J^c$ and $E \in I$. If every symmetrized two-particle rectangle in $\mathcal{M}_{\mathbf{x}}$ is (E,β)-nonresonant, then, in particular, every non-interactive symmetrized rectangle in $\mathcal{C}_{L,\ell}(\mathbf{x})$ is E-HNR and E-preregular, and hence $(m(\ell),\, E)$-regular by Lemma 4.24. As $m(\ell) > m_\ell$, we conclude that every non-interactive symmetrized box in $\mathcal{C}_{L,\ell}(\mathbf{x})$ is $(m_\ell,\, E)$-regular. Since $\boldsymbol{\omega} \in \mathcal{B}_J^c \cap \mathcal{A}^c$, $\mathcal{C}_{L,\ell}(\mathbf{x})$ contains at most $J-1$ pairwise ℓ-distant interactive $(m_\ell,\, E)$-nonregular boxes in $\mathcal{C}_{L,\ell}(\mathbf{x})$, and all other symmetrized boxes in $\mathcal{C}_{L,\ell}(\mathbf{x})$ are $(m_\ell,\, E)$-regular, it follows from Lemma 4.18 that $\mathbf{\Lambda}_{S,L}(\mathbf{x})$ is $(m_L,\, E)$-regular. If there exists a symmetrized two-particle rectangle in $\mathcal{M}_{\mathbf{x}}$ that is (E,β)-nonresonant, then every symmetrized two-particle rectangle in $\mathcal{M}_{\mathbf{y}}$ must be (E,β)-nonresonant, and thus $\mathbf{\Lambda}_{S,L}(\mathbf{y})$ is $(m_L,\, E)$-regular using the same argument as before. Thus for every $E \in I$ either $\mathbf{\Lambda}_{S,L}(\mathbf{x})$ is $(m_L,\, E)$-regular or $\mathbf{\Lambda}_{S,L}(\mathbf{y})$ is $(m_L,\, E)$-regular. It follows that

$$R\left(m_L,\, I,\, \mathbf{x},\, \mathbf{y},\, L\right) \subseteq \left(\mathcal{B}_J^c \cap \mathcal{A}^c \cap \mathcal{U}_J^c\right)^c, \tag{4.76}$$

so

$$\mathbb{P}\left\{R\left(m_L,\, I,\, \mathbf{x},\, \mathbf{y},\, L\right)\right\} \leq \mathbb{P}\left(\mathcal{B}_J\right) + \mathbb{P}\left(\mathcal{A}\right) + \mathbb{P}\left(\mathcal{U}_J\right). \tag{4.77}$$

Using independence and Lemma 4.22, we get

$$\mathbb{P}\left\{\mathcal{B}_J\right\} \leq 2\left(\tfrac{6L}{\ell}+2\right)^{4d} e^{-\frac{J}{2}L^{\frac{\zeta_2}{\gamma}}} \quad \text{and} \quad \mathbb{P}\left\{\mathcal{A}\right\} \leq 2\left(\tfrac{2L}{\ell}\right)^{2Nd} e^{-L^{\frac{\tau}{\gamma}}}. \tag{4.78}$$

We now fix

$$J \in \left(2L^{\beta-\frac{\zeta_2}{\gamma}}, 2L^{\beta-\frac{\zeta_2}{\gamma}}+2\right] \cap 2\mathbb{N},$$

so, for L sufficiently large, $\mathbb{P}\left\{\mathcal{B}_J\right\} \leq \frac{1}{3}e^{-L^{\zeta_2}}$, $\mathbb{P}\left\{\mathcal{A}\right\} \leq \frac{1}{3}e^{-L^{\zeta_2}}$, and $\mathbb{P}\left\{\mathcal{U}_J\right\} \leq \frac{1}{3}e^{-L^{\zeta_2}}$, and we conclude from (4.77) that

$$\mathbb{P}\left\{R\left(m_L,\, I,\, \mathbf{x},\, \mathbf{y},\, L\right)\right\} \leq e^{-L^{\zeta_2}}. \tag{4.79}$$

We now take L_0 large enough so that $m(L_0) > m_{L_0} = m_0$ and the above procedure can be carried out with $\ell = L_0$, $L_{k+1} = L_k^{\gamma}$ for $k = 0, 1, \ldots$, and $m_k \geq m_{k-1} - (2L_{k-1}^{-\kappa})$, where we write $m_k := m_{L_k}$. To finish the proof, we just need to make sure $m_k > \frac{m_0}{2}$ for all $k = 0, 1, \ldots$, which can be done taking L_0 sufficiently large, using the fact $m_0 \geq L_0^{-\kappa}$ as in [KlN1, Eq. 3.46].

□

4.2.7. *Completing the proof of the Bootstrap Multiscale Analysis for symmetrized two-particle boxes.* The proof of Theorem 2.9 follows from Propositions 4.15, 4.17, 4.19, 4.26 as in [GK1, Section 6], see also [KlN1, Section 3.5]. The result holds for all sufficiently large scales by the argument in [KlN1, Theorem 3.21].

5. Extracting dynamical localization from the Bootstrap Multiscale Analysis

In this section we prove Theorem 2.11. Here we present an improvement of [KlN1, Corollary 1.7], which we state using the Hausdorff distance dist_H in the n-particle setting. This result holds in the symmetrized distance if the conclusions of the Multiscale Analysis hold with respect to the dist_S.

Following the arguments in [KlN1, Section 4.1], we can prove that $H_{\boldsymbol{\omega}} = H_{\boldsymbol{\omega}}^{(n)}$ exhibits Anderson localization in the interval I, that is, for almost every $\boldsymbol{\omega}$, $\sigma\left(H_{\boldsymbol{\omega}}\right) \cap I = \sigma_{pp}\left(H_{\boldsymbol{\omega}}\right) \cap I$ and $\sigma_{cont}\left(H_{\boldsymbol{\omega}}\right) \cap I = \emptyset$. We fix $\nu = (nd+1)/2$ and for $\mathbf{a} \in \mathbb{Z}^{nd}$ let $T_{\mathbf{a}}$ denote the operator on $\ell^2(\mathbb{Z}^{nd})$ given by multiplication by $(1 + \|\mathbf{x} - \mathbf{a}\|^2)^{\nu/2}$. By $\chi_{\mathbf{x}}$ we denote the orthogonal projection onto $\delta_{\mathbf{x}}$. We will work with the following definition, from [GK5, Section 3],

Definition 5.1. *Given $\boldsymbol{\omega}, \lambda \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{Z}^{nd}$, define*

$$W_{\lambda,\boldsymbol{\omega}}(\mathbf{a}) := \begin{cases} \sup\limits_{\phi \in \mathcal{T}_{\lambda,\boldsymbol{\omega}}} \dfrac{\left\|\chi_{\mathbf{a}} \chi_{\{\lambda\}}(H_{\boldsymbol{\omega}})\phi\right\|}{\left\|T_{\mathbf{a}}^{-1} \chi_{\{\lambda\}}(H_{\boldsymbol{\omega}})\phi\right\|}, & \text{if } \chi_{\{\lambda\}}(H_{\boldsymbol{\omega}}) \neq 0 \\ 0, & \text{otherwise,} \end{cases} \tag{5.1}$$

where $\mathcal{T}_{\lambda,\boldsymbol{\omega}} = \{\phi \in \ell^2(\mathbb{Z}^{nd});\ \chi_{\{\lambda\}}(H_{\boldsymbol{\omega}})\phi \neq 0\}$, and

$$Z_{\lambda,\boldsymbol{\omega}}(\mathbf{a}) := \begin{cases} \dfrac{\left\|\chi_{\mathbf{a}} \chi_{\{\lambda\}}(H_{\boldsymbol{\omega}})\right\|_2}{\left\|T_{\mathbf{a}}^{-1} \chi_{\{\lambda\}}(H_{\boldsymbol{\omega}})\right\|_2}, & \text{if } \chi_{\{\lambda\}}(H_{\boldsymbol{\omega}}) \neq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

We have

$$Z_{\lambda,\boldsymbol{\omega}}(\mathbf{a}) \leq W_{\lambda,\boldsymbol{\omega}}(\mathbf{a}) \leq 1. \tag{5.3}$$

Theorem 5.2. *Let I be a closed interval where the conclusions of Theorem 2.9 hold for $H_{\boldsymbol{\omega}}$. Then, for every $\zeta \in (0,1)$ there exists a constant $C_\zeta > 0$ such that*

$$\begin{aligned} \mathbb{E}\left\{\sup_{|f|\leq 1} \left\|\chi_{\mathbf{x}}\, (f\,\chi_I)(H_{\boldsymbol{\omega}})\, \chi_{\mathbf{y}}\right\|_2^2\right\} &\leq \mathbb{E}\left\{\sup_{|f|\leq 1} \left\|\chi_{\mathbf{x}}\, (f\,\chi_I)(H_{\boldsymbol{\omega}})\, \chi_{\mathbf{y}}\right\|_1\right\} \\ &\leq C_\zeta e^{-d_H(\mathbf{x},\mathbf{y})^\zeta} \quad \text{for all} \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^{nd}, \end{aligned} \tag{5.4}$$

where the supremum is taken over all bounded Borel functions f on $\mathbb{R}$.

Proof. Since $H_{\boldsymbol{\omega}}$ exhibits Anderson localization in I for almost every $\boldsymbol{\omega}$, let $\{\psi_j(\boldsymbol{\omega})\}$ be an orthonormal base of the space $\operatorname{Ran} \chi_I(H_{\boldsymbol{\omega}})$ consisting of eigenfunctions of $H_{\boldsymbol{\omega}} \chi_I(H_{\boldsymbol{\omega}})$ with corresponding eigenvalues $\lambda_j(\boldsymbol{\omega})$. Then

$$H_{\boldsymbol{\omega}} \chi_I(H_{\boldsymbol{\omega}}) = \sum_j \lambda_j(\boldsymbol{\omega}) P_{\psi_j(\boldsymbol{\omega})}, \quad \text{and}$$

$$\sup_{|f|\leq 1} \left\|\chi_{\mathbf{x}}\, (f\,\chi_I)(H_{\boldsymbol{\omega}})\, \chi_{\mathbf{y}}\right\|_1 = \sup_{|f|\leq 1} \left\|\chi_{\mathbf{x}} \sum_j f(\lambda_j(\boldsymbol{\omega})) P_{\psi_j(\boldsymbol{\omega})}\, \chi_{\mathbf{y}}\right\|_1, \tag{5.5}$$

where $P_{\psi_j(\boldsymbol{\omega})}$ is the projection onto the space spanned by $\psi_j(\boldsymbol{\omega})$. Since $|f| \leq 1$, we have

$$\begin{aligned}\left\| \chi_{\mathbf{x}} \sum_j f(\lambda_j(\boldsymbol{\omega})) P_{\psi_j(\boldsymbol{\omega})} \chi_{\mathbf{y}} \right\|_1 &\leq \sum_j \left\| \chi_{\mathbf{x}} P_{\psi_j(\boldsymbol{\omega})} \chi_{\mathbf{y}} \right\|_1 \\ &\leq \sum_j \left\| \chi_{\mathbf{x}} P_{\psi_j(\boldsymbol{\omega})} \right\|_2 \left\| P_{\psi_j(\boldsymbol{\omega})} \chi_{\mathbf{y}} \right\|_2 . \end{aligned} \tag{5.6}$$

From Definition 5.1 and (5.3) we get

$$\begin{aligned} &\left\| \chi_{\mathbf{x}} P_{\psi_j(\boldsymbol{\omega})} \right\|_2 \leq Z_{\lambda_j,\boldsymbol{\omega}}(\mathbf{x}) \left\| T_{\mathbf{x}}^{-1} P_{\psi_j(\boldsymbol{\omega})} \right\|_2 \leq W_{\lambda_j,\boldsymbol{\omega}}(\mathbf{x}) \left\| T_{\mathbf{x}}^{-1} P_{\psi_j(\boldsymbol{\omega})} \right\|_2 \quad \text{and} \\ &\left\| \chi_{\mathbf{y}} P_{\psi_j(\boldsymbol{\omega})} \right\|_2 \leq W_{\lambda_j,\boldsymbol{\omega}}(\mathbf{y}) \left\| T_{\mathbf{y}}^{-1} P_{\psi_j(\boldsymbol{\omega})} \right\|_2 . \end{aligned} \tag{5.7}$$

Hence,

$$\begin{aligned} &\left\| \chi_{\mathbf{x}} \sum_j f(\lambda_j(\boldsymbol{\omega})) P_{\psi_j(\boldsymbol{\omega})} \chi_{\mathbf{y}} \right\|_1 \\ &\quad \leq \sum_j W_{\lambda_j,\boldsymbol{\omega}}(\mathbf{x}) \left\| T_{\mathbf{x}}^{-1} P_{\psi_j(\boldsymbol{\omega})} \right\|_2 W_{\lambda_j,\boldsymbol{\omega}}(\mathbf{y}) \left\| T_{\mathbf{y}}^{-1} P_{\psi_j(\boldsymbol{\omega})} \right\|_2 \\ &\quad \leq \left\| W_{\lambda_j,\boldsymbol{\omega}}(\mathbf{x}) W_{\lambda_j,\boldsymbol{\omega}}(\mathbf{y}) \right\|_{L^\infty(I, d\mu_{\boldsymbol{\omega}}(\lambda))} \sum_j \left\| T_{\mathbf{x}}^{-1} P_{\psi_j(\boldsymbol{\omega})} \right\|_2 \left\| T_{\mathbf{y}}^{-1} P_{\psi_j(\boldsymbol{\omega})} \right\|_2 . \end{aligned} \tag{5.8}$$

By the Cauchy-Schwarz inequality, we get

$$\begin{aligned} &\sum_j \left\| T_{\mathbf{x}}^{-1} P_{\psi_j(\boldsymbol{\omega})} \right\|_2 \left\| T_{\mathbf{y}}^{-1} P_{\psi_j(\boldsymbol{\omega})} \right\|_2 \\ &\qquad \leq \left(\sum_j \left\| T_{\mathbf{x}}^{-1} P_{\psi_j(\boldsymbol{\omega})} \right\|_2^2 \right)^{1/2} \left(\sum_j \left\| T_{\mathbf{y}}^{-1} P_{\psi_j(\boldsymbol{\omega})} \right\|_2^2 \right)^{1/2} . \end{aligned} \tag{5.9}$$

We have

$$\begin{aligned} &\sum_j \left\| T_{\mathbf{x}}^{-1} P_{\psi_j(\boldsymbol{\omega})} \right\|_2^2 \\ &\qquad = \sum_j \operatorname{tr} \left\{ T_{\mathbf{x}}^{-1} P_{\psi_j(\boldsymbol{\omega})}^2 T_{\mathbf{x}}^{-1} \right\} = \operatorname{tr} \left\{ T_{\mathbf{x}}^{-1} \chi_I(H_{\boldsymbol{\omega}}) T_{\mathbf{x}}^{-1} \right\} . \end{aligned} \tag{5.10}$$

Given a self-adjoint operator H on $\ell^2(\mathbb{Z}^{nd})$, we have (see [KlN1, Lemma 4.1] and its proof)

$$\operatorname{tr} \left\{ T_{\mathbf{x}}^{-1} \chi_I(H) T_{\mathbf{x}}^{-1} \right\} \leq C_{nd} < \infty \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^{nd}. \tag{5.11}$$

Thus we conclude that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{nd}$ and almost every $\boldsymbol{\omega}$ we have

$$\begin{aligned} &\left\| \chi_{\mathbf{x}} \sum_j f(\lambda_j(\boldsymbol{\omega})) P_{\psi_j(\boldsymbol{\omega})} \chi_{\mathbf{y}} \right\|_1 \\ &\qquad \leq C_{d,n} \left\| W_{\lambda_j,\boldsymbol{\omega}}(\mathbf{x}) W_{\lambda_j,\boldsymbol{\omega}}(\mathbf{y}) \right\|_{L^\infty(I, d\mu_{\boldsymbol{\omega}}(\lambda))} . \end{aligned} \tag{5.12}$$

To study $\left\|W_{\lambda_j,\boldsymbol{\omega}}(\mathbf{x})W_{\lambda_j,\boldsymbol{\omega}}(\mathbf{y})\right\|_{L^\infty(I,d\mu_{\boldsymbol{\omega}}(\lambda))}$, we divide it into two cases. For the first case, consider $d_H(\mathbf{x},\mathbf{y}) < L_\zeta$. Note that we always have

$$\|\chi_{\mathbf{a}}\psi\| \le \|\chi_{\mathbf{a}}T_{\mathbf{a}}\| \left\|T_{\mathbf{a}}^{-1}\psi\right\| \le \left\|T_{\mathbf{a}}^{-1}\psi\right\| \quad \text{for every} \quad \mathbf{a}\in\mathbb{R}^{nd}. \tag{5.13}$$

Hence,

$$\begin{aligned}\mathbb{E}\left(\|W_{\lambda,\boldsymbol{\omega}}(\mathbf{x})W_{\lambda,\boldsymbol{\omega}}(\mathbf{y})\|_{L^\infty(I,d\mu_{\boldsymbol{\omega}}(\lambda))}\right) &\le e^{d_H(\mathbf{x},\mathbf{y})^\zeta}e^{-d_H(\mathbf{x},\mathbf{y})^\zeta} \\ &\le e^{L_\zeta^\zeta}e^{-d_H(\mathbf{x},\mathbf{y})^\zeta} = C_\zeta e^{-d_H(\mathbf{x},\mathbf{y})^\zeta}.\end{aligned} \tag{5.14}$$

For the case $d_H(\mathbf{x},\mathbf{y}) = L \ge L_\zeta$, we consider the two cases: when $\boldsymbol{\omega}\in R(I,m,L,\mathbf{x},\mathbf{y})$ and when $\boldsymbol{\omega}\notin R(I,m,L,\mathbf{x},\mathbf{y})$ (recall (4.67)). By Equation (5.13), for every $\boldsymbol{\omega}\in R(I,m,L,\mathbf{x},\mathbf{y})$

$$\left|W_{\lambda_j,\boldsymbol{\omega}}(\mathbf{x})W_{\lambda_j,\boldsymbol{\omega}}(\mathbf{y})\right| \le 1 \quad \text{for} \quad \lambda_j\in I. \tag{5.15}$$

For the case that $\boldsymbol{\omega}\notin R(I,m,L,\mathbf{x},\mathbf{y})$ we have that for $E\in I$, $\boldsymbol{\Lambda}_{S;L}(\mathbf{x})$ or $\boldsymbol{\Lambda}_{S;L}(\mathbf{y})$ is (m,E)-Sregular. Without loss of generality, say $\boldsymbol{\Lambda}_L(\mathbf{x})$ is (m,E)-regular. Suppose $H\psi = E\psi$ with $\|\psi\|=1$, then (see [KlN1, Eq. 4.8])

$$\|\chi_{\mathbf{x}}\psi\| \le e^{-mL/2}\left\|\chi_{\partial_+\boldsymbol{\Lambda}_L(\mathbf{x})}\psi\right\|, \tag{5.16}$$

while

$$\left\|\chi_{\partial_+\boldsymbol{\Lambda}_L(\mathbf{x})}\psi\right\| \le \left\|\chi_{\partial_+\boldsymbol{\Lambda}_L(\mathbf{x})}T_{\mathbf{x}}\right\|\left\|T_{\mathbf{x}}^{-1}\psi\right\|. \tag{5.17}$$

Since

$$\left\|\chi_{\partial_+\boldsymbol{\Lambda}_L(\mathbf{a})}T_{\mathbf{a}}\right\| \le L^{2\nu} \quad \text{for every} \quad \mathbf{a}\in\mathbb{R}^{nd}, \tag{5.18}$$

we obtain,

$$\|\chi_{\mathbf{x}}\psi\| \le e^{-mL/2}L^{2\nu}\left\|T_{\mathbf{x}}^{-1}\psi\right\| \tag{5.19}$$

Thus, for $\boldsymbol{\omega}\notin R(I,m,L,\mathbf{x},\mathbf{y})$, we get

$$\left|W_{\lambda_j,\boldsymbol{\omega}}(\mathbf{x})W_{\lambda_j,\boldsymbol{\omega}}(\mathbf{y})\right| \le e^{-mL/2}L^{2\nu} \quad \text{for} \quad \lambda_j\in I. \tag{5.20}$$

We can decompose the expectation in two parts corresponding to the set of $\omega\in R(I,m,L,\mathbf{x},\mathbf{y})$ and its complement, and obtain

$$\begin{aligned}&\mathbb{E}\left(\|W_{\lambda,\boldsymbol{\omega}}(\mathbf{x})W_{\lambda,\boldsymbol{\omega}}(\mathbf{y})\|_{L^\infty(I,d\mu_{\boldsymbol{\omega}}(\lambda))}\right) \\ &\qquad \le L^{2\nu}e^{-mL/2} + e^{-L^\zeta} \le e^{-L^{\zeta'}} = e^{-(d_H(\mathbf{x},\mathbf{y}))^{\zeta'}},\end{aligned} \tag{5.21}$$

provided L_ζ is large enough and $\zeta' < \zeta$. Hence, we get

$$\begin{aligned}\mathbb{E}\left\{\sup_{|f|\le 1}\|\chi_{\mathbf{x}}(f\chi_I)(H_{\boldsymbol{\omega}})\chi_{\mathbf{y}}\|_2^2\right\} &\le \mathbb{E}\left\{\sup_{|f|\le 1}\|\chi_{\mathbf{x}}(f\chi_I)(H_{\boldsymbol{\omega}})\chi_{\mathbf{y}}\|_1\right\} \\ &\le C_\zeta e^{-d_H(\mathbf{x},\mathbf{y})^\zeta} \quad \text{for all} \quad \mathbf{x},\mathbf{y}\in\mathbb{R}^{nd}.\end{aligned} \tag{5.22}$$

□

Corollary 5.3. *Let I be a closed interval where the conclusions of Theorem 2.9 hold for $H_{\boldsymbol{\omega}}$. Then*

$$\mathbb{E}\left\{\sup_{|f|\le 1}\left\|\langle d_H(\mathbf{X},\mathbf{y})\rangle^{\frac{p}{2}}(f\chi_I)(H_{\boldsymbol{\omega}})\chi_{\mathbf{y}}\right\|_2^2\right\} < \infty \quad \textit{for all} \quad \mathbf{y}\in\mathbb{R}^{nd}, \tag{5.23}$$

where the supremum is taken over all bounded Borel functions f on $\mathbb{R}$.

Proof.

$$\begin{aligned}
&\mathbb{E}\left\{\sup_{|f|\le 1}\left\|\langle d_H(\mathbf{X},\mathbf{y})\rangle^{\frac{p}{2}}\,(f\,\chi_I)(H_{\boldsymbol\omega})\,\chi_{\mathbf{y}}\right\|_2^2\right\}\\
&\quad\le C_1\sum_{\mathbf{x}\in\mathbb{Z}^{nd}}\langle d_H(\mathbf{x},\mathbf{y})\rangle^p\,\mathbb{E}\left\{\sup_{|f|\le 1}\left\|\chi_{\mathbf{x}}f\left(H_{\boldsymbol\omega}\right)\chi_I\left(H_{\boldsymbol\omega}\right)\chi_{\mathbf{y}}\right\|_2^2\right\}\\
&\quad\le C_1\sum_{\mathbf{x}\in\mathbb{Z}^{nd}}\langle d_H(\mathbf{x},\mathbf{y})\rangle^p\,\mathbb{E}\left\{\sup_{|f|\le 1}\left\|\chi_{\mathbf{x}}f\left(H_{\boldsymbol\omega}\right)\chi_I\left(H_{\boldsymbol\omega}\right)\chi_{\mathbf{y}}\right\|_1\right\} \qquad (5.24)\\
&\quad\le C_2\sum_{\mathbf{x}\in\mathbb{Z}^{nd}}\langle d_H(\mathbf{x},\mathbf{y})\rangle^p\,e^{-d_H(\mathbf{x},\mathbf{y})^\zeta}.
\end{aligned}$$

The result follows from the fact $\sum_{\mathbf{x}\in\mathbb{Z}^{nd}}\langle d_H(\mathbf{x},\mathbf{y})\rangle^p\,e^{-d_H(\mathbf{x},\mathbf{y})^\zeta}<\infty$ shown in [AW1, Lemma A.3].

□

Appendix A. Wegner estimates

Theorem A.1. *Let $H_{\boldsymbol\omega}^{(n)}$ be the n-particle Anderson model. Consider $\boldsymbol\Lambda\subset\mathbb{Z}^{nd}$ such that:*

(i) *If $\sharp=\infty,S$, $\boldsymbol\Lambda=\boldsymbol\Lambda^{(n)}_{\sharp;\boldsymbol L}(\mathbf{x})$ is an n-particle $\sharp$-rectangle of center $\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{R}^{nd}$ and sides $\boldsymbol L=(L_1,L_2,\ldots,L_n)\in[1,\infty)^n$, take $\Gamma=\Lambda_{L_k}(x_k)$ for some $k\in\{1,...,n\}$, and set $L=\max\{L_1,L_2,\ldots,L_n\}$.*

(ii) *If $\sharp=H$, $\boldsymbol\Lambda=\boldsymbol\Lambda^{(n)}_{H;L}(\mathbf{x})\subset\mathbb{Z}^{nd}$ is an n-particle H-box of center $\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{R}^{nd}$ and side $L\ge 1$, and take $\Gamma=\bigcup_{k=1}^n\Lambda_L(x_k)$.*

Then, for any interval $I\subset\mathbb{R}$ we have

$$\mathbb{E}_\Gamma\left(\mathrm{tr}\chi_I(H_\omega)\right)\le C_n^{(\sharp)}\left\|\rho\right\|_\infty L^{nd}, \qquad (A.1)$$

where $C_n^{(\infty)}=n$, $C_n^{(S)}=n(n!)$, and $C_n^{(H)}=n^{2n+1}$. In particular, for any $E\in\mathbb{R}$ and $\varepsilon>0$, we have

$$\mathbb{P}_\Gamma\left(\|G_{\boldsymbol\Lambda}(E)\|\ge\tfrac{1}{\varepsilon}\right)=\mathbb{P}_\Gamma\Big\{d\left(\sigma(H_{\boldsymbol\Lambda}),E\right)\le\varepsilon\Big\}\le 2C_n^{(\sharp)}\left\|\rho\right\|_\infty\varepsilon L^{nd}. \qquad (A.2)$$

For $\sharp=\infty$ this is [KlN1, Theorem 2.3], whose proof can be modified to give the result for $\sharp=S,H$.

Appendix B. Combes-Thomas estimate for restrictions of discrete Schrödinger operators to arbitrary subsets

For convenience we state and prove a Comes-Thomas estimate for restrictions of discrete Schrödinger operators to arbitrary subsets (cf. [GK2, K1]).

Theorem B.1. *Let $H=-\Delta+V$ be a discrete Schrödinger operator on $\ell^2(\mathbb{Z}^d)$, where Δ is the centered Laplacian operator. Given $S\subset\mathbb{Z}^d$, let H_S be the restriction of H onto S with Dirichlet boundary condition. Then for every $z\notin\sigma(H_S)$, setting $\eta_z=\mathrm{dist}\,(z,\sigma(H))$, for all $\varepsilon\in(0,1)$ we get*

$$\left|\left\langle\delta_x,(H_S-z)^{-1}\delta_y\right\rangle\right|\le\tfrac{1}{\eta_z(1-\varepsilon)}e^{-\log\left(\frac{\varepsilon\eta_z}{2d}+1\right)\|y-x\|}\quad\textit{for all}\quad x,y\in S. \qquad (B.1)$$

Proof. Given $v = (v_1, \dots, v_d) \in \mathbb{Z}^d$, let M_v be the multiplication operator given by the function $e^{v \cdot x}$ and U_v be the multiplication operator given by the function $e^{-v \cdot x}$ on $\ell^2(\mathbb{Z}^d)$. We set

$$H_v = M_v H_S U_v = M_v(-\Delta_S + V_S)U_v = M_v(-\Delta_S)U_v + V_S. \tag{B.2}$$

Let $\{e_1, \dots, e_d\}$ be the standard basis for $\mathbb{R}^d$, and set

$$e_{d+j} = -e_j \quad \text{for} \quad j = 1, \dots, d. \tag{B.3}$$

Given $\psi \in \ell^2(\mathbb{Z}^d)$, we have

$$\begin{aligned}(-\Delta_S U_v \psi)(x) &= - \sum_{y \in S;\, |y-x|_1 = 1} e^{-v \cdot y} \psi(y) \\ &= - \sum_{j=1,\dots,2d;\, x+e_j \in S} e^{-v\cdot(x+e_j)} \psi(x+e_j) = - \sum_{j=1,\dots,2d;\, x+e_j \in S} e^{-v\cdot x} e^{-v \cdot e_j} \psi(x+e_j).\end{aligned}$$

Hence,

$$\begin{aligned}(M_v(-\Delta_S)U_v\psi)(x) &= - \sum_{j=1,\dots,2d;\, x+e_j \in S} e^{v\cdot x} e^{-v\cdot x} e^{-v\cdot e_j} \psi(x+e_j) \\ &= - \sum_{j=1,\dots,2d;\, x+e_j \in S} e^{-v\cdot e_j} \psi(x+e_j) = - \sum_{j=1,\dots,2d;\, x+e_j \in S} \left(e^{-v\cdot e_j} - 1 + 1\right) \psi(x+e_j) \\ &= -\Delta_S \psi(x) - \sum_{j=1,\dots,2d;\, x+e_j \in S} \left(e^{-v\cdot e_j} - 1\right) \psi(x+e_j).\end{aligned} \tag{B.4}$$

Let us define the operator B_v by

$$(B_v\psi)(x) = \sum_{j=1,\dots,2d;\, x+e_j \in S} \left(e^{-v\cdot e_j} - 1\right) \psi(x+e_j). \tag{B.5}$$

Then

$$\begin{aligned}|(B_v\psi)(x)| &= \left| \sum_{j=1,\dots,2d;\, x+e_j \in S} \left(e^{-v\cdot e_j} - 1\right) \psi(x+e_j) \right| \\ &\le 2d \max_{j=1,\dots,2d} \left\{ e^{|v_j|} - 1 \right\} \|\psi\| \le 2d(e^{\|v\|} - 1) \|\psi\|,\end{aligned}$$

so $\|B_v\| \le 2d(e^{\|v\|} - 1)$.

Thus, for $z \notin \sigma(H_S)$, letting $\eta_z = \operatorname{dist}(z, \sigma(H_S))$, and requiring

$$\|B_v\| \left\| (H_S - z)^{-1} \right\| < 1, \tag{B.6}$$

we get

$$(H_v - z)^{-1} = (H_S - B_v - z)^{-1} = (H_S - z)^{-1} \sum_{k=0}^{\infty} \left(B_v (H_S - z)^{-1}\right)^k, \tag{B.7}$$

and hence

$$\left\| (H_v - z)^{-1} \right\| \le \tfrac{1}{\eta_z} \tfrac{1}{1 - \frac{\|B_v\|}{\eta_z}} = \tfrac{1}{\eta_z - \|B_v\|}. \tag{B.8}$$

If we take $\|B_v\| \le \varepsilon \eta_z$ for some $\varepsilon < 1$, which can be achieved by requiring $2d(e^{|v|} - 1) \le \varepsilon \eta_z$, then (B.6) is satisfied and

$$\left\| (H_v - z)^{-1} \right\| \le \tfrac{1}{\eta_z - \|B_v\|} \le \tfrac{1}{\eta_z (1-\varepsilon)}. \tag{B.9}$$

Moreover, setting $d_z = \log\left(\frac{\varepsilon\eta_z}{2d}+1\right)$, we get

$$2d(e^{\|v\|}-1) \le \varepsilon\eta_z \iff \|v\| \le \log\left(\tfrac{\varepsilon\eta_z}{2d}+1\right) = d_z. \tag{B.10}$$

Thus, for v satisfying (B.10), we have

$$\begin{aligned}
&\left|\left\langle \delta_x, (H_S - z)^{-1}\delta_y\right\rangle\right| = \left|\left\langle \delta_x, M_v U_v (H_S - z)^{-1} M_v U_v \delta_y\right\rangle\right| \qquad \text{(B.11)}\\
&= \left|\left\langle U_v\delta_x, U_v (H_S - z)^{-1} M_v U_v \delta_y\right\rangle\right| = \left|\left\langle e^{-v\cdot x}\delta_x, (H_v - z)^{-1} e^{-v\cdot y}\delta_y\right\rangle\right|\\
&= e^{-v\cdot(y-x)}\left|\left\langle \delta_x, (H_v - z)^{-1}\delta_y\right\rangle\right| \le e^{-v\cdot(y-x)}\left\|(H_v - z)^{-1}\right\|\\
&\le e^{-v\cdot(y-x)}\tfrac{1}{\eta_z(1-\varepsilon)} \quad \text{for all} \quad x,y \in S.
\end{aligned}$$

Choosing $v = d_z \frac{y-x}{\|y-x\|}$, we get

$$\left|\left\langle \delta_x, (H_S - z)^{-1}\delta_y\right\rangle\right| \le \tfrac{1}{\eta_z(1-\varepsilon)} e^{-d_z\|y-x\|} = \tfrac{1}{\eta_z(1-\varepsilon)} e^{-\log\left(\frac{\varepsilon\eta_z}{2d}+1\right)\|y-x\|}. \tag{B.12}$$

□

Appendix C. Discrete version of an auxiliary result in [GK4]

The following is a generalization of [GK4, Lemma 6.4] to the discrete setting, stated in Lemma 3.2.

Lemma C.1. *Let $H_{\boldsymbol{\omega}}^{(n)}$ be a random n-particle Schrödinger operator satisfying a Wegner estimate for $\sharp$-boxes of the form (A.1) in an open interval $\mathcal{I}$. Let us denote by $\boldsymbol{\Lambda}$ be the n-particle $\sharp$-box $\boldsymbol{\Lambda}_{\sharp;L}^{(n)}$, where $\sharp \in \{\infty, S, H\}$. Let $p_0 > 0$ and $\gamma > nd$. There exists a scale $\mathcal{L} = \mathcal{L}(\gamma, n, d, \rho, p_0)$ such that, given $E \in \mathcal{I}$, $L \ge \mathcal{L}$, and subsets $B_1, B_2 \subset \boldsymbol{\Lambda}$ such that $\partial_-\boldsymbol{\Lambda} \subset B_2$, for each $a > 0$ and $\varepsilon > 0$ we have, for $\mathbf{u} \in B_1$ and $\mathbf{y} \in B_2$*

$$\begin{aligned}
&\mathbb{P}\left(a < |G_{\boldsymbol{\Lambda}}(E + i\varepsilon; \mathbf{y}, \mathbf{u})|\right)\\
&\qquad \le \frac{4L^{\gamma+2nd}}{a} \sup_{\mathbf{k}\in\partial_+\boldsymbol{\Lambda}\cup B_2} \mathbb{E}\left(|G(E+i\varepsilon; \mathbf{k}, \mathbf{u})|\right) + p_0
\end{aligned} \tag{C.1}$$

and

$$\begin{aligned}
&\mathbb{P}\left(a < |G_{\boldsymbol{\Lambda}}(E; \mathbf{y}, \mathbf{u})|\right)\\
&\quad \le \frac{8L^{\gamma+2nd}}{a} \sup_{\mathbf{k}\in\partial_+\boldsymbol{\Lambda}\cup B_2} \mathbb{E}\left(|G(E+i\varepsilon; \mathbf{k}, \mathbf{u})|\right) + 2^{3/2} C_n^{(\sharp)} \|\rho\|_\infty \sqrt{\frac{\varepsilon}{a}} L^{nd} + p_0.
\end{aligned} \tag{C.2}$$

Proof. Note that there exists a positive constant $C(n,d)$ such that for $L > C(n,d)$ we have $|\partial_+\boldsymbol{\Lambda}| \le L^{nd}$ for all $\sharp \in \{\infty, S, H\}$, and center $\mathbf{x} \in \mathbb{R}^{nd}$.

We write $z := E + i\varepsilon \in \mathbb{C}$. We use the geometric resolvent identity (see [K1, Section 5]) to get

$$G_{\boldsymbol{\Lambda}}(z; \mathbf{y}, \mathbf{u}) = G(z; \mathbf{y}, \mathbf{u}) - \sum_{(\mathbf{k}',\mathbf{k})\in\partial\boldsymbol{\Lambda}} G(z; \mathbf{k}, \mathbf{u}) G_{\boldsymbol{\Lambda}}(z; \mathbf{y}, \mathbf{k}') \tag{C.3}$$

where $(\mathbf{k}', \mathbf{k}) \in \partial\boldsymbol{\Lambda}$ means $\mathbf{k}' \in \partial_-\boldsymbol{\Lambda}$ and $\mathbf{k} \in \partial_+\boldsymbol{\Lambda}$, with $\|\mathbf{k} - \mathbf{k}'\|_1 = 1$.

From this, we obtain

$$|G_{\boldsymbol{\Lambda}}(z; \mathbf{y}, \mathbf{u})| \le |G(z; \mathbf{y}, \mathbf{u})| + \|G_{\boldsymbol{\Lambda}}(E)\| \sum_{\mathbf{k}\in\partial_+\boldsymbol{\Lambda}} |G(z; \mathbf{k}, \mathbf{u})|. \tag{C.4}$$

Then,

$$\mathbb{P}\left(a < |G_{\mathbf{\Lambda}}(z;\mathbf{y},\mathbf{u})|\right) \leq \mathbb{P}\left(\frac{a}{2} < |G(z;\mathbf{y},\mathbf{u})|\right) + \mathbb{P}\left(\frac{a}{2} < \|G_{\mathbf{\Lambda}}(E)\| \sum_{\mathbf{k}\in\partial_+\mathbf{\Lambda}} |G(z;\mathbf{k},\mathbf{u})|\right). \tag{C.5}$$

We bound the second term in the r.h.s. as follows,

$$\mathbb{P}\left(\frac{a}{2} < \|G_{\mathbf{\Lambda}}(E)\| \sum_{\mathbf{k}\in\partial_+\mathbf{\Lambda}} |G(z;\mathbf{k},\mathbf{u})|\right) \leq \mathbb{P}\left(\frac{a}{2L^{\gamma}} < \sum_{\mathbf{k}\in\partial_+\mathbf{\Lambda}} |G(z;\mathbf{k},\mathbf{u})|\right) + \mathbb{P}\left(L^{\gamma} < \|G_{\mathbf{\Lambda}}(E)\|\right). \tag{C.6}$$

Note that

$$\mathbb{P}\left(\frac{a}{2L^{\gamma}} < \sum_{\mathbf{k}\in\partial_+\mathbf{\Lambda}} |G(z;\mathbf{k},\mathbf{u})|\right) \leq |\partial_+\mathbf{\Lambda}| \sup_{\mathbf{k}\in\partial_+\mathbf{\Lambda}} \mathbb{P}\left(\frac{a}{2L^{\gamma}\,|\partial_+\mathbf{\Lambda}|} < |G(z;\mathbf{k},\mathbf{u})|\right). \tag{C.7}$$

We use this, Chebyshev's inequality and the Wegner estimate (A.1) to bound (C.5) and obtain

$$\begin{aligned}\mathbb{P}\left(a < |G_{\mathbf{\Lambda}}(z;\mathbf{y},\mathbf{u})|\right) &\leq \frac{2}{a}\mathbb{E}\left(|G(z;\mathbf{y},\mathbf{u})|\right) + \frac{2L^{\gamma}\,|\partial_+\mathbf{\Lambda}|^2}{a} \sup_{\mathbf{k}\in\partial_+\mathbf{\Lambda}} \mathbb{E}\left(|G(z;\mathbf{k},\mathbf{u})|\right) \\ &\quad + 2C_n^{(\sharp)}\,\|\rho\|_{\infty}\, L^{-\gamma+nd}\end{aligned} \tag{C.8}$$

We will take $L > \mathcal{L}_1$ for a $\mathcal{L}_1 = \mathcal{L}_1(n,d)$ such that $|\partial_+\mathbf{\Lambda}| < L^{nd}$. We will, furthermore, take $L > \mathcal{L}_2 = \mathcal{L}_2(\gamma,n,d,\rho,p_0)$ such that the last term in (C.8) $2C_n^{(\sharp)}\,\|\rho\|_{\infty}\, L^{-\gamma+nd} < p_0$.

$$\mathbb{P}\left(a < |G_{\mathbf{\Lambda}}(z;\mathbf{y},\mathbf{u})|\right) \tag{C.9}$$

$$\leq \frac{2}{a}\mathbb{E}\left(|G(z;\mathbf{y},\mathbf{u})|\right) + \frac{2L^{\gamma+2nd}}{a} \sup_{\mathbf{k}\in\partial_+\mathbf{\Lambda}} \mathbb{E}\left(|G(z;\mathbf{k},\mathbf{u})|\right) + p_0. \tag{C.10}$$

We will give a common bound for the first two terms in the r.h.s., which yields, for $L \geq \mathcal{L}_0 = \max\{\mathcal{L}_1, \mathcal{L}_2, 1\}$ depending on γ, n, d, ρ and p_0,

$$\mathbb{P}\left(a < |G_{\mathbf{\Lambda}}(z;\mathbf{y},\mathbf{u})|\right) \leq \frac{4L^{\gamma+2nd}}{a} \sup_{k\in\partial_+\mathbf{\Lambda}\cup B_2} \mathbb{E}\left(|G(z;\mathbf{k},\mathbf{u})|\right) + p_0. \tag{C.11}$$

Next, the resolvent identity gives

$$G_{\mathbf{\Lambda}}(E;\mathbf{y},\mathbf{u}) = G_{\mathbf{\Lambda}}(E+i\varepsilon;\mathbf{y},\mathbf{u}) + i\varepsilon G_{\mathbf{\Lambda}}(E)G_{\mathbf{\Lambda}}(E+i\varepsilon)(\mathbf{y},\mathbf{u}), \tag{C.12}$$

thus,

$$|G_{\mathbf{\Lambda}}(E;\mathbf{y},\mathbf{u})| \leq |G_{\mathbf{\Lambda}}(E+i\varepsilon;\mathbf{y},\mathbf{u})| + \varepsilon\,\|G_{\mathbf{\Lambda}}(E)\|^2. \tag{C.13}$$

Finally,

$$
\begin{aligned}
&\mathbb{P}\left(a < |G_{\mathbf{\Lambda}}(E;\mathbf{y},\mathbf{u})|\right)\\
&\quad \le \mathbb{P}\left(\frac{a}{2} < |G_{\mathbf{\Lambda}}(E+i\varepsilon;\mathbf{y},\mathbf{u})|\right) + \mathbb{P}\left(\frac{a}{2\varepsilon} < \|G_{\mathbf{\Lambda}}(E)\|^2\right) \qquad \text{(C.14)}\\
&\qquad \le \frac{8L^{\gamma+2nd}}{a} \sup_{\mathbf{k}\in\partial_+\mathbf{\Lambda}\cup B_2} \mathbb{E}\left(|G(E+i\varepsilon;\mathbf{k},\mathbf{u})|\right) + 2^{3/2} C_n^{(\sharp)} \|\rho\|_\infty \sqrt{\frac{\varepsilon}{a}} L^{nd} + p_0.
\end{aligned}
$$

$\square$

References

[ASW] Aizenman, M., Sims, R., Warzel, S.: Stability of the absolutely continuous spectrum of random Schrödinger operators on tree graphs. Prob. Theor. Rel. Fields, **136**, 363-394 (2006)

[AW1] Aizenman, M., Warzel, S.: Localization bounds for multiparticle systems. Commun. Math. Phys. **290**, 903-934 (2009)

[AW2] Aizenman, M., Warzel, S.: Resonant delocalization for random Schrödinger operators on trees. J. Eur. Math. Soc. (JEMS) **15**, 1167-1222 (2013)

[BCSS] Boutet de Monvel, A., Chulaevsky, V., Stollmann, P., Suhov, Y.: Wegner-type bounds for a multi-particle continuous Anderson model with an alloy-type external potential. J. Stat. Phys. **138**, 553-566 (2010)

[CL] Carmona, R, Lacroix, J.: Spectral Theory of Random Schrödinger Operators. Birkhäuser, Boston (1990)

[Ch1] Chulaevsky, V.: Optimized estimates of the regularity of the conditional distribution of the sample mean. Mathematics and Statistics **3**, 46-52 (2015)

[ChBS] Chulaevsky, V., Boutet de Monvel, A., Suhov, Y.: Dynamical localization for a multi-particle model with an alloy-type external random potential. Nonlinearity, **24** , 1451-1472 (2011)

[ChS1] Chulaevsky, V., Suhov, Y.: Wegner bounds for a two particle tight binding model. Commun. Math. Phys. **283**, 479-489 (2008)

[ChS2] Chulaevsky, V., Suhov, Y.: Eigenfunctions in a two-particle Anderson tight binding model. Commun. Math. Phys. **289**, 701-723 (2009)

[ChS3] Chulaevsky, V., Suhov, Y.: Multi-particle Anderson Localization: Induction on the number of particles. Math. Phys. Anal. Geom., no. 2, 117-139 (2009)

[ChS4] Chulaevsky, V., Suhov, Y.: Efficient Anderson localization bounds for large multi-particle systems. Journal of Spectral Theory, in press. Preprint arXiv:1404.3978

[FW] Fauser, M., Warzel, S.: Multiparticle localization for disordered systems on continuous space via the fractional moment method, Rev. Math. Phys. **27**, 1550010 (2015)

[FrHS] Froese, R., Hasler, D., Spitzer, W.: Absolutely continuous spectrum for the Anderson Model on a tree: A geometric proof of Klein's Theorem. Commun. Math. Phys., **269**, 239-257 (2007)

[GK1] Germinet, F., Klein, A.: Bootstrap multiscale Analysis and Localization in Random Media. Commun. Math. Phys. **222**, 415-448 (2001)

[GK2] Germinet, F., Klein, A.: Decay of operator-valued kernels of functions of Schrödinger and other operators. Proc. Amer. Math. Soc. **131**, 911-920 (2003)

[GK4] Germinet, F., Klein, A.: A characterization of the Anderson metal-insulator transport transition. Duke Math. J. **124**, 309-351 (2004)

[GK5] Germinet, F., Klein, A.: New characterization of the region of complete localization for random Schrödinger operators. J. Stat. Phys. **122**, 73-94 (2006)

[GK6] Germinet, F., Klein, A.: A comprehensive proof of localization for continuous Anderson models with singular random potentials. J. Eur. Math. Soc. **15**, 53-143 (2013)

[GKS] Germinet, F., Klein, A., Schenker, J. H.:Dynamical delocalization in random Landau Hamiltonians. Ann. of Math. **166**, 215-244 (2007)

[K1] Kirsch, W.: An invitation to random Schrödinger operators. In *Random Schrödinger Operators*. Panor. Synthèses **25**, Soc. Math. France, Paris, 1-119 (2008)

[K2] Kirsch, W.: A Wegner estimate for multi-particle random Hamiltonians. Zh. Mat. Fiz. Anal. Geom. **4**, 121–127 (2008)

[KM] Kirsch, W., Martinelli, F. : On the spectrum of Schrödinger operators with a random potential. Commun. Math. Phys. **85**, 329-350 (1982)

[Kl1] Klein,A.: Absolutely continuous spectrum in the Anderson model on the Bethe lattice, Math. Res. Lett. **1**, 399-407 (1994)

[Kl2] Klein, A.: Extended states in the Anderson model on the Bethe lattice. Adv. Math. **133**, 163-184 (1998)

[KlN1] Klein, A., Nguyen, S: The bootstrap multiscale analysis for the multi-particle Anderson model. J. Stat. Phys. **151**, 983-973 (2013)

[KlN2] Klein, A., Nguyen, S: Bootstrap multiscale analysis and localization for multi-particle continuous Anderson Hamiltonians. J. Spectr. Theory **5**, 399-444 (2015)

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