

# THE $\kappa^+$ -ANTICHAIN PROPERTY FOR $(\kappa, 1)$ -SIMPLIFIED MORASSES

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ABSTRACT. We discuss Shelah, Väänänen and Veličković's recently introduced  $\kappa^+$ -antichain property for  $(\kappa, 1)$ -simplified morasses. We give a streamlined characterization of the property, and show how the property can be destroyed by forcing and hence that it is consistent that no  $(\omega, 1)$ -simplified morasses have the property. We briefly touch on the combination of the antichain property with complete amalgamation systems.

## PRELIMINARIES

We start by fixing some standard notation and reminding the reader of the definition of  $(\kappa, 1)$ -simplified morasses, in order to be able to introduce an extremely interesting partial order on  $\kappa^+$ , compatible with the usual ordering of the ordinals, recently isolated by Shelah, Väänänen and Veličković ([5]).

**Notation 1.1.** For a set  $X$  and cardinal  $\kappa$ ,  $[X]^\kappa = \{Y \subseteq X \mid \overline{Y} = \kappa\}$ .

**Notation 1.2.** If  $\tau < \theta$  are ordinals the set of order preserving functions from  $\tau$  to  $\theta$ ,  $\{f \mid f : \tau \rightarrow_{o.p.} \theta\}$ , is denoted  $(\theta)^\tau$ .

**Notation 1.3.** The *strong supremum* of a set of ordinals  $X$ ,  $\text{ssup}(X)$ , is the least  $\gamma$  such that  $X \subseteq \gamma$ . So  $\text{ssup}(X)$  is  $\max(X) + 1$  if  $X$  has a maximal element and  $\text{sup}(X)$  otherwise.

**Notation 1.4.** If  $\kappa$  is a regular cardinal then  $\text{Add}(\kappa, 1)$  is the usual forcing to add a single Cohen subset of  $\kappa$ : the set of partial functions of size  $< \kappa$  from  $\kappa$  to 2 ordered by reverse inclusion.

**Definition 1.5.** If  $\tau < \theta$  are ordinals then  $\mathcal{F} = \{\text{id}, h\} \subseteq (\theta)^\tau$  is an *amalgamation pair* if there is some  $\sigma < \tau$  such that

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- $h \upharpoonright \sigma = \text{id}$ ,
- for all  $\xi$  such that  $\sigma + \xi < \tau$  we have  $h(\sigma + \xi) = \tau + \xi$ , and
- $\tau \cup h''\tau$  is an initial segment of  $\theta$ .

In this case we say that  $\sigma$  is the *splitting point* of  $\mathcal{F}$ . We say  $\mathcal{F}$  is *exact* if  $\theta = \tau \cup h''\tau$  and *almost exact* if  $\theta = (\tau \cup h''\tau) + 1$ .

Let  $\kappa$  be a regular cardinal.

**Definition 1.6.** ([6])  $\mathcal{M} = \langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa \rangle \rangle$  is a  $(\kappa, 1)$ -*simplified morass* if

- $\langle \theta_\alpha \mid i < \kappa \rangle \in (\kappa)^\kappa$  and  $\theta_\kappa = \kappa^+$
- for each  $\alpha \leq \beta \leq \kappa$  one has that  $\mathcal{F}_{\alpha\beta} \subseteq \{f \mid f : \theta_\alpha \rightarrow_{o.p.} \theta_\beta\}$  and

- for each  $\alpha \leq \kappa$ ,  $\mathcal{F}_{\alpha\alpha} = \{\text{id}\}$

- for each  $\alpha < \kappa$ ,

$\mathcal{F}_{\alpha\alpha+1}$  is a singleton or an amalgamation pair

- for each  $\alpha \leq \beta \leq \gamma \leq \kappa$ ,

$$\mathcal{F}_{\alpha\gamma} = \{g \cdot f \mid f \in \mathcal{F}_{\alpha\beta} \& g \in \mathcal{F}_{\beta\gamma}\}$$

- for each  $\alpha \leq \beta < \kappa$ ,  $\overline{\mathcal{F}_{\alpha\beta}} < \kappa$

- if  $\varepsilon \leq \kappa$  is a limit ordinal then  $\mathcal{M}$  is *directed at*  $\varepsilon$ :

if  $\alpha, \beta < \varepsilon$ ,  $e_\alpha \in \mathcal{F}_{\alpha\varepsilon}$  and  $e_\beta \in \mathcal{F}_{\beta\varepsilon}$  there are  $\gamma \in [\alpha \cup \beta, \varepsilon)$ ,  $g \in \mathcal{F}_{\gamma\varepsilon}$ ,  $f_\alpha \in \mathcal{F}_{\alpha\gamma}$  and  $f_\beta \in \mathcal{F}_{\beta\gamma}$  such that  $e_\alpha = g \cdot f_\alpha$  and  $e_\beta = g \cdot f_\beta$

- $\bigcup \{f''\theta_\alpha \mid \alpha < \kappa \& f \in \mathcal{F}_{\alpha\kappa}\} = \kappa^+$ .

We say  $\mathcal{M}$  is *neat* if  $\theta_\beta = \bigcup \{f''\theta_\alpha \mid f \in \mathcal{F}_{\alpha\beta}\}$  for all  $\alpha < \beta < \kappa$  (and  $\theta_0 = 1$ ).

From now on let  $\mathcal{M} = \langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa \rangle \rangle$  be a  $(\kappa, 1)$ -simplified morass.

Recall Stanley's important lemma.

**Lemma 1.7.** ([6], Stanley). *If  $\alpha \leq \beta \leq \kappa$ ,  $\xi_0, \xi_1 < \theta_\alpha$  and  $\xi < \theta_\beta$ ,  $f_0, f_1 \in \mathcal{F}_{\alpha\beta}$ , and  $f(\xi_0) = f(\xi_1) = \xi$ , then  $\xi_0 = \xi_1$  and  $f_0 \upharpoonright \xi_0 = f_1 \upharpoonright \xi_0$ .*

This motivates the following two pieces of notation.

**Notation 1.8.** (*cf.* [2], [3]) For  $\alpha \leq \kappa$  write  $\xi_\alpha$  for the unique  $\xi'$  such that there is some  $f \in \mathcal{F}_{\alpha\kappa}$  with  $f(\xi') = \xi$ , and write  $\psi_\xi^\alpha$  for  $f \upharpoonright \xi_\alpha$ . (In [3] the extended notation,  $\psi_{(\alpha, \xi_\alpha), (\kappa, \xi)}$ , was also used for  $\psi_\xi^\alpha$ .)

**Notation 1.9.** If  $\alpha \leq \beta \leq \kappa$ ,  $\xi < \theta_\alpha$  and  $f \in \mathcal{F}_{\alpha\beta}$  then  $(\alpha, \xi) \triangleleft (\beta, f(\xi))$ .

By Stanley's lemma  $\triangleleft$  is a (collection of) tree(s).

It is also useful to have a name for the function that marks at which level below  $\kappa$  two elements of  $\kappa^+$  separate (*i.e.*, *branch apart*).

**Notation 1.10.** ([2]) If  $\xi, \zeta < \kappa$  then  $b(\xi, \zeta) =$  the least  $\alpha + 1$  such that  $\xi_{\alpha+1} \neq \zeta_{\alpha+1}$ .

## 2. $\prec$ AND THE $\kappa^+$ -ANTICHAIN PROPERTY

We continue to let  $\mathcal{M} = \langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa \rangle \rangle$  be a  $(\kappa, 1)$ -simplified morass.

Shelah, Väänänen and Veličković ([5]) isolated a very interesting partial order, compatible with the usual ordering of the ordinals, on  $\kappa^+$ .

**Definition 2.1.** ([5]) For  $\xi, \zeta < \kappa^+$  define  $\xi \prec \zeta$  if and only if  $\xi_\alpha \leq \zeta_\alpha$  for all  $\alpha \leq \kappa$  for which both  $\xi_\alpha$  and  $\zeta_\alpha$  are defined.

Note that  $\xi_\kappa = \xi$  for all  $\xi < \kappa^+$ , so  $\xi \prec \zeta$  implies  $\xi < \zeta$ .

**Definition 2.2.**  $\mathcal{M}$  has the  $\kappa^+$ -*antichain property* if for every  $X \in [\kappa^+]^{\kappa^+}$  there are  $\xi, \zeta \in X$  such that  $\xi \prec \zeta$ .

Shelah, Väänänen and Veličković showed in [5] that the usual forcing for adding an  $(\omega_1, 1)$ -simplified morass (as in [6], [10]) actually adds one with the  $\omega_2$ -antichain property. Their proof, in fact, applies just as well for arbitrary regular  $\kappa$  in place of  $\omega_1$ .

Shelah, Väänänen and Veličković's original definition of the antichain property (which they gave in the case  $\kappa = \omega_1$ ) in fact talks about an order induced by  $\prec$  on sequences of ordinals.

**Definition 2.3.**  $\mathcal{M}$  has the *SVV- $\kappa^+$ -antichain property* if for every  $\delta < \kappa$  and  $X \in [(\kappa^+)^\delta]^{\kappa^+}$  there are  $s, t \in X$  such that  $\text{dom}(s) = \text{dom}(t) = \delta$  and for all  $\gamma < \delta$  we have  $s(\gamma) \prec t(\gamma)$

**Note 2.4.**  $\mathcal{M}$  has the *SVV- $\kappa^+$ -antichain property* if and only if for every  $X \in [(\kappa^+)^{<\kappa}]^{\kappa^+}$  there are  $s, t \in X$  such that  $\text{dom}(s) = \text{dom}(t)$  and for all  $\tau \in \text{dom}(s)$  we have  $s(\tau) \prec t(\tau)$ .

*Proof.* Immediate

★2.4

**Proposition 2.5.** *If  $\mathcal{M}$  has the SVV- $\kappa^+$ -antichain property it has the  $\kappa^+$ -antichain property.*

*Proof.* We prove the contrapositive. If  $X \in [\kappa^+]^{\kappa^+}$  and  $\delta < \kappa$ , by thinning if necessary we may assume that if  $\xi, \zeta \in X$  and  $\xi < \zeta$  then  $\xi + \delta < \zeta$ . For  $\xi \in X$  let  $s^\xi \in (\kappa^+)^{<\kappa}$  be the function given by  $s^\xi(\gamma) = \xi + \gamma$ . If  $X$  is an antichain in  $\prec$  then  $\{s^\xi \mid \xi \in X\}$  is an antichain in the product ordering.

★2.5

In [5] the SVV- $\omega_2$ -antichain property is always proved, mentioned or used in the context of CH holding. The culmination of the next few results, in the case  $\kappa = \omega_1$ , indicates that this is not mere coincidence.

**Notation 2.6.** For each  $\alpha < \kappa$  and  $\xi < \theta_\alpha$  let

$$B_{(\alpha, \xi)} = \{f(\xi) \mid f \in \mathcal{F}_{\alpha\kappa}\}.$$

As a mnemonic,  $B_{(\alpha, \xi)}$  is the *blossom* above  $(\alpha, \xi)$ .

**Notation 2.7.** Let

$$F = \{\xi < \kappa^+ \mid \forall \alpha < \kappa (\xi_\alpha \text{ is defined} \longrightarrow \overline{\overline{B_{(\alpha, \xi_\alpha)}}} = \kappa^+)\}.$$

So  $F$  is the set of elements of  $\kappa^+$  all of whose predecessors in  $\triangleleft$  have a maximal cardinality collection of blossom above them.

**Lemma 2.8.**  *$F$  is cobounded in  $\kappa^+$ , i.e.,  $\overline{\overline{\kappa^+ \setminus F}} \leq \kappa$ .*

*Proof.* For each  $\xi \in \kappa^+ \setminus F$  let  $\alpha(\xi)$  be the least  $\alpha$  such that  $\overline{\overline{B_{(\alpha, \xi_\alpha)}}} < \kappa^+$ . As  $\mathcal{M}$  is a  $(\kappa, 1)$ -simplified morass we have that  $\theta_\beta < \kappa$  for all  $\beta < \kappa$ . Hence  $\overline{\overline{\{(\alpha(\xi), \xi_{\alpha(\xi)}) \mid \xi \in \kappa^+ \setminus F\}}} \leq \kappa$ . Thus we have that  $\kappa^+ \setminus F \subseteq \bigcup \{B_{(\alpha(\xi), \xi_{\alpha(\xi)})} \mid \xi \in \kappa^+ \setminus F\}$ . However the latter is a union of  $\kappa$  many sets of size  $\kappa$ .

★2.8

**Lemma 2.9.** *If  $\xi \in F$  there are unboundedly many  $\alpha < \kappa$  such that  $\mathcal{F}_{\alpha\alpha+1}$  is an amalgamation pair and  $\sigma_\alpha \leq \xi_\alpha = \xi_{\alpha+1}$ .*

*Proof.* Let  $\xi \in F$ . Suppose, towards a contradiction that the lemma is false for  $\xi$ . Let  $\beta < \kappa$  be least such that for all  $\alpha \in [\beta, \kappa)$  it is not the case that  $\mathcal{F}_{\alpha+1}$  is an amalgamation pair and  $\sigma_\alpha \leq \xi_\alpha = \xi_{\alpha+1}$ . Then  $B_{(\beta, \xi_\beta)} \subseteq \xi + 1$ , contradicting the assumption that  $\xi \in F$  (and hence  $\overline{B_{(\beta, \xi_\beta)}} = \kappa^+$ ). ★2.9

**Corollary 2.10.** *There are antichains of size  $\kappa$  in  $\prec \upharpoonright \kappa \times \kappa$ . There are antichains of order-type  $\kappa + 1$  in  $\prec$ .*

*Proof.* Let  $\xi \in F \setminus \kappa$ . Let  $H_\xi = \{\alpha < \kappa \mid \mathcal{F}_{\alpha+1} \text{ is an amalgamation pair and } \sigma_\alpha \leq \xi_\alpha = \xi_{\alpha+1}\}$ . For each  $\alpha \in H_\xi$  we have  $\mathcal{F}_{\alpha+1} = \{\text{id}, h_\alpha\}$ . Then  $\{h_\alpha(\xi_\alpha) \mid \alpha \in H_\xi\}$  is an antichain in  $\prec$ , and we still have an antichain if we adjoin  $\xi$  to this set. ★2.10

**Proposition 2.11.** *If  $\kappa < 2^{<\kappa}$  then no  $(\kappa, 1)$ -simplified morass  $\mathcal{M}$  has the SVV- $\kappa^+$ -antichain property.*

*Proof.* Let  $\delta < \kappa$  be such that  $\kappa^+ \leq \kappa < 2^{<\delta}$ . By Corollary (2.10) choose a  $\prec$ -antichain  $\{\xi^i \mid i < \delta\}$  of size  $\delta$  with  $\xi^i < \xi^j$  for  $i < j$ .

Now let  $X$  consist of all distinct increasing sequences of length  $\delta$  from  $\{\xi^i \mid i < \delta\}$ . Suppose that  $s, t \in X$ . Let  $\gamma < \delta$  be least such that  $s(\gamma) \neq t(\gamma)$ . We have that there are distinct  $i, j < \delta$  such that  $s(\gamma) = \xi^i$  and  $t(\gamma) = \xi^j$ . As  $\{\xi^i \mid i < \delta\}$  is a  $\prec$ -antichain we have  $s(\gamma) \not\prec t(\gamma)$  and  $t(\gamma) \not\prec s(\gamma)$ . Thus the SVV- $\kappa^+$ -antichain property fails. ★2.11

Consequently the following result, together with Proposition (2.5), shows that the property defined in Definition (2.2) is a streamlined equivalent of the SVV- $\kappa^+$ -antichain property when the latter does not simply always fail for cardinal arithmetic reasons. This explains taking Definition (2.2) as our official definition of *the  $\kappa^+$ -antichain property*.

**Proposition 2.12.** *If  $2^{<\kappa} = \kappa$  and  $\mathcal{M}$  has the  $\kappa^+$ -antichain property then  $\mathcal{M}$  has the SVV- $\kappa^+$ -antichain property.*

*Proof.* Again we prove the contrapositive.

Suppose that  $X \in [(\kappa^+)^\delta]^{\kappa^+}$ . As  $2^{<\kappa} = \kappa$ , after thinning if necessary, we can suppose that we can enumerate  $X$  as  $\langle s^i \mid i < \kappa^+ \rangle$  with there being some  $\rho < \kappa$  such that  $s^i \upharpoonright \rho = s^j \upharpoonright \rho$  and  $\text{ssup}(\text{rge}(s^i)) < s^j(\rho)$  for  $i < j < \kappa^+$ .

For  $i < \kappa^+$  let  $\zeta^i = \text{ssup}(\text{rge}(s^i))$ . By thinning again (if necessary), again using  $2^{<\kappa} = \kappa$ , we may assume that there is some  $\alpha < \kappa$ , some

$s \in (\theta_\alpha)^{\delta+1}$  and some maps  $f^i \in \mathcal{F}_{\alpha\kappa}$  for  $i < \kappa^+$ , such that for all  $i < \kappa^+$  we have  $s^i(\gamma) = f^i(s(\gamma))$  for all  $\gamma < \delta$  and  $f^i(s(\delta)) = \zeta^i$ .

Suppose that  $i < j < \kappa^+$  and  $s^i \not\prec s^j$ . Let  $\beta \in [\alpha, \kappa)$  be least such that there is  $\gamma < \kappa$  with  $(s^j(\gamma))_{\beta+1} < (s^i(\gamma))_{\beta+1}$ . Then we have that

$$\sigma_\beta \leq (s^i(\gamma))_\beta \leq (s^j(\gamma))_\beta = (s^j(\gamma))_{\beta+1} < \zeta_{\beta+1}^j < \theta_\beta \leq (s^i(\gamma))_{\beta+1} < \zeta_{\beta+1}^i.$$

Hence we have shown that if  $X$  is an antichain of size  $\kappa^+$  in the product ordering derived from  $\prec$  then  $\{\zeta^j \mid j < \kappa^+\}$  is an antichain in  $\prec$ .

★2.12

### 3. DESTROYING THE ANTICHAIN PROPERTY

It is worthwhile observing that there are no long chains in  $\prec$ .

**Proposition 3.1.** *There are no  $\prec$ -chains of length  $\kappa^+$ .*

*Proof.* For  $s, t \in (\kappa)^\kappa$  set  $s < t$  if  $s(\alpha) \leq t(\alpha)$  for all  $\alpha < \kappa$  and there is some  $\alpha < \kappa$  such that  $s(\alpha) < t(\alpha)$ . It is well known that there are no (strictly increasing) chains of length  $\kappa^+$  in  $((\kappa)^\kappa, <)$ .

(To see the latter, if  $\langle s^i \mid i < \kappa^+ \rangle$  enumerates a strictly increasing chain in  $<$ , in increasing order then set  $I_0 = \kappa^+$ , and, by induction on  $\beta \leq \kappa$  set  $I_\beta = \bigcap_{\alpha < \beta} \{i \in I_\alpha \mid s^i(\alpha) = \tau_\alpha\}$  and  $\tau_\beta = \max(\{s^i(\beta) \mid i \in I_\beta\})$ . Then  $I_\kappa$  has size  $\kappa^+$  and for all  $i \in I_\kappa$  and  $\alpha < \kappa$  we have  $s^i(\alpha) = \tau_\alpha$ . This is a contradiction to the chain being increasing.)

In order to prove the proposition, now suppose, towards a contradiction, that  $\langle \xi^i \mid i < \kappa^+ \rangle$  is a  $\prec$ -chain of length  $\kappa^+$ . By thinning, if necessary, suppose that there is some  $\alpha^* < \kappa$  such that for all  $i < \kappa^+$  we have that  $\alpha^*$  is the least  $\alpha$  such that  $\xi_\alpha^i$  is defined. Apply the result that there are no  $\kappa^+$ -chains of length  $\kappa^+$  in  $((\kappa)^\kappa, <)$  to  $\langle s^i \mid i < \kappa^+ \rangle$  where for  $i < \kappa^+$  and  $\alpha \in [\alpha^*, \kappa)$  we take  $s^i(\alpha) = \xi_\alpha^i$ .<sup>1</sup> ★3.1

<sup>1</sup>*Morass-y version of the same proof.* Suppose, towards a contradiction, that  $X \in [\kappa^+]^{\kappa^+}$  is a  $\prec$ -chain. For  $\alpha < \kappa$  let  $X_\alpha = \{\xi \in X \mid \xi_{\alpha+1} < \theta_\alpha\}$ . For each  $\alpha < \kappa$  we have that  $X_\alpha$  is an initial segment of  $X$ , and hence either  $\overline{X_\alpha} \leq \kappa$  or  $X_\alpha = X$ . Set  $Y = X \setminus \bigcup \{X_\alpha \mid \alpha < \kappa \text{ \& } \overline{X_\alpha} \leq \kappa\}$ . Thus  $\overline{Y} = \kappa^+$ .

Now let  $\xi, \zeta \in Y$  and set  $\alpha = b(\xi, \zeta)$ , so that  $\xi_\alpha = \zeta_\alpha$  and  $\xi_{\alpha+1} \neq \zeta_{\alpha+1}$ . We derive a contradiction. If  $\overline{X_\alpha} \leq \kappa$  then we would have that  $\xi, \zeta \notin X_\alpha$  and hence  $\theta_\alpha \leq \xi_{\alpha+1}, \zeta_{\alpha+1}$ , and so  $\xi_{\alpha+1} = h_\alpha(\xi_\alpha) = h_\alpha(\zeta_\alpha) = \zeta_{\alpha+1}$ , and thus  $b(\xi, \zeta) \neq \alpha$ . On the other hand, if  $X_\alpha = X$  then  $\xi_{\alpha+1}, \zeta_{\alpha+1} < \theta_\alpha$  and hence, again  $\xi_{\alpha+1} = \xi_\alpha = \zeta_\alpha = \zeta_{\alpha+1}$ , and so  $b(\xi, \zeta) \neq \alpha$ . ★3.1

**Corollary 3.2.** *For each  $\delta < \kappa$  there are no  $\prec$ -chains of length  $\kappa^+$  in the product order on  $(\kappa^+)^\delta$ .*

However we can actually prove a stronger result. In order to state it we need to make a definition.

**Definition 3.3.** Let  $\mathbb{P}_\mathcal{M}$  be the forcing consisting of conditions which are small antichains in  $\prec$ : so  $p \in \mathbb{P}_\mathcal{M}$  if  $p \in [\kappa^+]^{<\kappa}$  and for all  $\xi, \zeta \in p$  we have  $\xi \not\prec \zeta$ , ordered by  $q \leq p$  if  $p \subseteq q$ .

Observe that with this definition we can restate Proposition (3.1) as that if  $\mathfrak{X} \in [\mathbb{P}_\mathcal{M}]^{\kappa^+}$  and for all  $p \in \mathfrak{X}$  we have  $\bar{p} = 1$  then  $\mathfrak{X}$  is not an antichain in  $\mathbb{P}_\mathcal{M}$ .

**Proposition 3.4.** *Suppose  $2^{<\kappa} = \kappa$ . If  $\mathcal{M}$  has the  $\kappa^+$ -antichain property then  $\mathbb{P}_\mathcal{M}$  has the  $\kappa^+$ -chain condition.*

*Proof.* Let  $\mathfrak{A} \in [\mathbb{P}_\mathcal{M}]^{\kappa^+}$  be an antichain and assume, after thinning if necessary, that the set forms a  $\Delta$ -system with root  $a$ , with  $\text{ssup}(p) < \min(q \setminus a)$  or vice versa for each pair  $p, q \in \mathfrak{A}$ .

Let  $N \prec H_{\kappa^{++}}$  be an elementary submodel of size  $\kappa$  with  $N \cap \kappa^+ = \delta \in \kappa^+$ ,  $P, \mathfrak{A}, \mathcal{M} \in N$  and  $a, N^{<\kappa} \subseteq N$ . Then  $\mathcal{M} \cap N = \langle \langle \theta_\alpha \mid \alpha < \kappa \rangle, \langle \mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta < \kappa \rangle \rangle \frown \langle \delta, \langle \{f \in \mathcal{F}_{\alpha\kappa} \mid \text{rge}(f) \subseteq \delta\} \mid \alpha < \kappa \rangle \rangle$ . Thus if  $\xi < \delta$ ,  $\alpha < \kappa$  and  $\tau < \theta_\alpha$  then  $N \models \text{“}\xi_\alpha = \tau\text{”}$  if and only if  $\xi_\alpha = \tau$ .

Choose  $p, q \in \mathfrak{A} \setminus N$  with  $\delta \leq \min(p \setminus a)$  and  $\text{ssup}(p) < \min(q \setminus a)$ . Let  $\beta^* =$  the least  $\alpha < \kappa$  such that there is some map  $f \in \mathcal{F}_{\alpha\kappa}$  with  $p \cup q \subseteq \text{rge}(f)$ . Note that if  $\xi \in p$  and  $\zeta \in q \setminus a$  there is some  $\alpha \leq \beta^*$  such that  $\xi_\alpha < \zeta_\alpha$ .

Let  $\{q(\gamma) \mid \gamma < \varepsilon\}$  enumerate  $q$  in increasing order. For  $\gamma < \varepsilon$  and  $\alpha \leq \beta^*$  set  $y(\gamma, \alpha) = q(\gamma)_\alpha$ .

By elementarity and the closure of  $N$  there is some  $r \in \mathfrak{A} \cap N$  such that, letting  $\{r(\gamma) \mid \gamma < \varepsilon\}$  enumerate  $r$  in increasing order, for all  $\gamma < \varepsilon$  and  $\alpha \leq \beta^*$  we have  $r(\gamma)_\alpha = y(\gamma, \alpha)$ .

But then we have that if  $\xi = r(\gamma) \in r$  and  $\zeta \in p \setminus a$  there is some  $\alpha \leq \beta^*$  such that  $q(\gamma)_\alpha < \zeta_\alpha$ , and hence  $\xi_\alpha = r(\gamma)_\alpha = y(\gamma, \alpha) = q(\gamma)_\alpha < \zeta_\alpha$ ; and as  $r \subseteq \delta$  and  $\delta \leq \min(p \setminus a)$  there is some  $\alpha < \kappa$  such that  $\xi_\alpha < \zeta_\alpha$ .

Thus  $p \cup r$  is a  $\prec$ -antichain and hence is a condition in  $\mathbb{P}_{\mathcal{M}}$ . However we then have that  $p \cup r \leq p, r$ , thus contradicting  $\mathfrak{A}$  being an antichain in  $\mathbb{P}_{\mathcal{M}}$ . ★3.4

Observe that there is some freedom in the argument above: if we also ‘reflect’  $p$  to a condition  $s$  in  $N$ , so that – writing informally – we have  $s \ll r \ll p \ll q$ , rather than amalgamating  $r$  and  $p$  we could instead amalgamate  $s$  and  $q$ .

**Proposition 3.5.** *Suppose  $2^{<\kappa} = \kappa$ . If  $V \models$  “ $\mathcal{M}$  has the  $\kappa^+$ -antichain property” and  $G$  is  $\mathbb{P}_{\mathcal{M}}$ -generic over  $V$  then  $\text{Card}^{V[G]} = \text{Card}^V$  and*

$$V[G] \Vdash \text{“} \mathcal{M} \text{ does not have the } \kappa^+ \text{-antichain property. ”}$$

*Proof.* The antichain property is destroyed as  $\mathbb{P}_{\mathcal{M}}$  generically adds an antichain of length  $\kappa^+$ . Cardinals are preserved since  $\mathbb{P}_{\mathcal{M}}$  has the  $\kappa^+$ -chain condition and is  $\kappa$ -closed. ★3.5

Let us focus briefly on the case  $\kappa = \omega$ . Recall Velleman’s theorem ([8]) that ZFC implies there are always  $(\omega, 1)$ -simplified morasses. In contrast we have the following regarding simplified morasses with the  $\omega_1$ -antichain property.

**Corollary 3.6.**  *$MA_{\omega_1}$  implies no  $(\omega, 1)$ -simplified morass has the  $\omega_1$ -antichain property.*

*Proof.* Suppose  $\mathcal{M}$  is an  $(\omega, 1)$ -simplified morass with the  $\omega_1$ -antichain property. By Proposition (3.4) in the case  $\kappa = \omega$  there is there is a ccc forcing to destroy the property and so, applying  $MA_{\omega_1}$ ,  $\mathcal{M}$  does not have the  $\omega_1$ -antichain property – a contradiction. ★3.6

Unfortunately one cannot directly generalize Corollary (3.6) to higher cardinals and obtaining a similar independence result. The Appendix of [4] gives examples showing that no generalization of Martin’s Axiom for forcings with the  $\kappa^+$ -cc or strengthenings of it can hold for collections of forcings which would include  $\mathbb{P}_{\mathcal{M}}$ . Those results do not preclude that one could, in principle, iterate this specific forcing in order to reach a model in which no  $(\kappa, 1)$ -simplified morass has the  $\kappa^+$ -antichain property, however we are not aware of any applicable iteration theorems.



One might wonder whether there is a simpler forcing notion which destroys the antichain property, which one could use instead of  $\mathbb{P}_{\mathcal{M}}$ , in the hope of side-stepping these difficulties. However there are severe inherent difficulties with such a plan.

**Definition 3.7.** A forcing notion  $P$  has the  $\kappa^+$ -Knaster property if given any  $X \in [P]^{\kappa^+}$  there is some  $Z \in [X]^{\kappa^+}$  such that any two elements of  $Z$  are compatible.

**Proposition 3.8.** *If  $\mathcal{M}$  has the  $\kappa^+$ -antichain property and  $\mathbb{P}$  has the  $\kappa^+$ -Knaster property then  $\Vdash_{\mathbb{P}}$  “  $\mathcal{M}$  has the  $\kappa^+$ -antichain property ”*

*Proof.* Suppose that  $\Vdash_{\mathbb{P}}$  “  $\dot{\mathcal{A}}$  is an antichain of size  $\kappa^+$  ”. Let  $p_i \Vdash_{\mathbb{P}}$  “  $\xi^i \in \dot{\mathcal{A}}$  ” and  $\xi^i \in p_i$  for  $i < \kappa^+$  and a strictly increasing sequence  $\langle \xi^i \mid i < \kappa^+ \rangle$ . By the Knaster property let  $I \in [\kappa^+]^{\kappa^+}$  be such that for  $i, j \in I$  we have that  $p_i$  and  $p_j$  are compatible; for such  $i, j$  let  $p_{ij} \leq p_i, p_j$ . Then, for  $i, j \in I$  we have  $p_{ij} \Vdash_{\mathbb{P}}$  “  $\xi^i \not\leq \xi^j$  ”, and hence  $\xi^i \not\leq \xi^j$ . So  $\langle \xi^i \mid i < \kappa^+ \rangle$  is an antichain of size  $\kappa^+$  in the ground model. \*3.8

Clearly the forcing  $\mathbb{P}_{\mathcal{M}}$  used in Proposition (3.4) does not have the  $\kappa^+$ -Knaster property, but we are not aware of any iteration technology which works successfully for iterands of this type.

#### 4. COMPLETE AMALGAMATION SYSTEMS AND THE ANTICHAIN PROPERTY

We make a couple of remarks about a strengthening of the notion of a complete amalgamation system ([9]) and the  $\kappa^+$ -antichain property.

**Definition 4.1.** ([9]) Let  $\langle \langle \rho_\alpha, X_\alpha, Y_\alpha \rangle \mid \alpha < \kappa \rangle$  be a sequence of triples where  $\rho_\alpha < \kappa$  and  $X_\alpha, Y_\alpha \subseteq \theta_\alpha$  for each  $\alpha < \kappa$ . Define, by induction on  $\alpha \leq \kappa$ ,

$$\begin{aligned} A_0 &= \emptyset, \\ A_{\alpha+1} &= \{ \langle \rho, f \text{“} X, f \text{“} Y \rangle \mid f \in \mathcal{F}_{\alpha+1} \ \& \ \langle \rho, X, Y \rangle \in A_\alpha \} \cup \\ &\quad \{ \langle \rho_\alpha, X_\alpha, h \text{“} Y_\alpha \rangle \}, \text{ and} \\ A_\lambda &= \{ \langle \rho, f \text{“} X, f \text{“} Y \rangle \mid \exists \alpha < \lambda \ f \in \mathcal{F}_{\alpha\lambda} \ \& \ \langle \rho, X, Y \rangle \in A_\alpha \} \\ &\quad \text{for limit } \lambda \leq \kappa. \end{aligned}$$

The sequence  $\langle \langle \rho_\alpha, X_\alpha, Y_\alpha \rangle \mid \alpha < \kappa \rangle$  is an *amalgamation system* if for all  $\alpha < \kappa$  either  $X_\alpha = Y_\alpha$  or  $\langle \rho_\alpha, X_\alpha, Y_\alpha \rangle \in A_\alpha$  or  $\langle \rho_\alpha, Y_\alpha, X_\alpha \rangle \in A_\alpha$ .

(New in this paper and *not* taken from [9].) It is a *strong amalgamation system* if for all  $\alpha < \kappa$  either  $X_\alpha = Y_\alpha$  or  $\langle \rho_\alpha, X_\alpha, Y_\alpha \rangle \in A_\alpha$ .

It is *complete* if in addition whenever  $\rho < \kappa$  and  $\mathfrak{X} \in [[\kappa^+]^{<\kappa}]^{\kappa^+}$  there are distinct  $X, Y \in \mathfrak{X}$  such that  $\langle \rho, X, Y \rangle \in A_\kappa$ .

**Lemma 4.2.** *If  $\langle \langle \rho_\alpha, X_\alpha, Y_\alpha \rangle \mid \alpha < \kappa \rangle$  is a complete amalgamation system then  $2^{<\kappa} = \kappa$ .*

*Proof.* Let  $\lambda < \kappa$ . For each  $X \subseteq \lambda$  let  $\mathfrak{X}_X = \{X \cup \{\lambda, \tau\} \mid \tau \in (\lambda, \kappa^+)\}$ . By the completeness of the amalgamation system there is some  $\alpha < \kappa$  and  $f \in \mathcal{F}_{\alpha\kappa}$  such that  $X_\alpha = \overline{X} \cup \{\overline{\lambda}, \overline{\tau}\}$ ,  $\overline{X} \subseteq \overline{\lambda} < \overline{\tau} < \theta_\alpha$ ,  $X = f^{\overline{X}}$ ,  $f(\overline{\lambda}) = \lambda$  and  $f(\overline{\tau}) = \tau$ . Thus  $\mathcal{P}(\lambda) \subseteq \{\psi_\lambda^\alpha(X_\alpha \cap \overline{\lambda}) \mid \alpha < \kappa\}$ . \*4.2

**Theorem 4.3.** ([9]). *If  $\kappa = \mu^+$  and  $2^\mu = \kappa$  there is a complete amalgamation system for every  $(\kappa, 1)$ -simplified morass.*

**Corollary 4.4.** *If  $\kappa = \mu^+$  and  $\mathcal{M}$  is a  $(\kappa, 1)$ -simplified morass there is a complete amalgamation system for  $\mathcal{M}$  if and only if  $2^\mu = \kappa$ .*

**Theorem 4.5.** ([9]) *If  $V \models 2^{<\kappa} = \kappa$  and  $c$  is  $\text{Add}(\kappa, 1)$ -generic over  $V$  there is a complete amalgamation system for every  $(\kappa, 1)$ -simplified morass in  $V[c]$ . (Of course, as  $(\kappa^+)^V = (\kappa^+)^{V[c]}$ , every  $(\kappa, 1)$ -simplified morass in  $V$  remains such in  $V[c]$ .)*

**Proposition 4.6.** *If there is a complete strong amalgamation system for  $\mathcal{M}$  then  $\mathcal{M}$  satisfies the  $\kappa^+$ -antichain property.*

*Proof.* Immediate from the definitions. \*4.6

**Proposition 4.7.** *If  $\kappa = \mu^+$  and  $2^\mu = \kappa$  there is a complete strong amalgamation system for every  $(\kappa, 1)$ -simplified morass which satisfies the  $\kappa^+$ -antichain property. If  $V \models 2^{<\kappa} = \kappa$  and  $c$  is  $\text{Add}(\kappa, 1)$ -generic over  $V$  there is a complete strong amalgamation system for every  $(\kappa, 1)$ -simplified morass in  $V[c]$  with the  $\kappa^+$ -antichain property.*

*Proof.* Exactly as Velleman's proofs, but using the  $\kappa^+$ -antichain property to ensure that one can choose strong amalgamation systems. \*4.7

Note that if  $V \models 2^{<\kappa} = \kappa$  and  $c$  is  $\text{Add}(\kappa, 1)$ -generic over  $V$  then every  $(\kappa, 1)$ -simplified morass in  $V$  with the  $\kappa^+$ -antichain property continues to have the  $\kappa^+$ -antichain property in  $V[c]$  because  $\text{Add}(\kappa, 1)$  trivially has the  $\kappa^+$ -Knaster condition.

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