

# ASCENDING PATHS AND FORCINGS THAT SPECIALIZE HIGHER ARONSZAJN TREES

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ABSTRACT. In this paper, we study trees of uncountable regular heights containing ascending paths of small width. This combinatorial property of trees generalizes the concept of a cofinal branch and it causes trees to be non-special not only in  $V$ , but also in every cofinality-preserving outer model of  $V$ . Moreover, under certain cardinal arithmetic assumptions, the non-existence of such paths through a tree turns out to be equivalent to the statement that the given tree is special in a cofinality preserving forcing extension of the ground model. We will present a number of consistency results on the non-existence of trees without cofinal branches containing ascending paths of small width. In contrast, we will construct such trees using certain combinatorial principles.

As an application of our results, we show that the consistency strength of a potential forcing axiom for  $\sigma$ -closed, well-met partial orders satisfying the  $\aleph_2$ -chain condition and collections of  $\aleph_2$ -many dense subsets is at least a weakly compact cardinal. In addition, we will use our results to show that the infinite productivity of the Knaster property characterizes weak compactness in canonical inner models. Finally, we study the influence of the Proper Forcing Axiom on trees containing ascending paths.

## 1. INTRODUCTION

The purpose of this paper is to study combinatorial properties of trees of uncountable regular heights that cause these trees to be non-special in a very absolute way. Remember that a partial order  $\mathbb{T}$  is a *tree* if it has a unique minimal element  $\text{root}(\mathbb{T})$  and sets of the form  $\text{pred}_{\mathbb{T}}(t) = \{s \in \mathbb{T} \mid s <_{\mathbb{T}} t\}$  are well-ordered by  $<_{\mathbb{T}}$  for every  $t \in \mathbb{T}$ . Given a tree  $\mathbb{T}$  and  $t \in \mathbb{T}$ , we define  $\text{lh}_{\mathbb{T}}(t)$  to be the order-type of  $(\text{pred}_{\mathbb{T}}(t), <_{\mathbb{T}})$  and we define  $\text{ht}(\mathbb{T}) = \sup_{t \in \mathbb{T}} \text{lh}_{\mathbb{T}}(t)$  to be the *height* of  $\mathbb{T}$ . Moreover, we define  $\mathbb{T}(\gamma) = \{t \in \mathbb{T} \mid \text{lh}_{\mathbb{T}}(t) = \gamma\}$  and  $\mathbb{T}_{<\gamma} = \{t \in \mathbb{T} \mid \text{lh}_{\mathbb{T}}(t) < \gamma\}$  for every tree  $\mathbb{T}$  and  $\gamma < \text{ht}(\mathbb{T})$ . Finally, given a subset  $S$  of  $\text{ht}(\mathbb{T})$ , we define  $\mathbb{T} \upharpoonright S$  to be the suborder of  $\mathbb{T}$  whose underlying set is the set  $\bigcup \{\mathbb{T}(\gamma) \mid \gamma \in S\}$ .

One of the most basic questions about trees of infinite heights is the question of the existence of *cofinal branches*, i.e. the existence of a subset  $B$  of the tree  $\mathbb{T}$  such that  $B$  is linearly ordered by  $<_{\mathbb{T}}$  and the set  $\{\text{lh}_{\mathbb{T}}(t) \mid t \in B\}$  is unbounded in  $\text{ht}(\mathbb{T})$ . As phrased by Todorćević in the introduction of [26, Section 6.1], it turns out that

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a large class of trees of uncountable regular heights have no cofinal branches for *very special reasons*.

In the remainder of this paper,  $\theta$  will denote an uncountable regular cardinal and, unless otherwise noted,  $\mathbb{T}$  will denote a tree of height  $\theta$ .

**Definition 1.1** (Todorčević). Let  $S$  be a subset of  $\theta$ .

- (i) A map  $r : \mathbb{T} \upharpoonright S \rightarrow \mathbb{T}$  is *regressive* if  $r(t) <_{\mathbb{T}} t$  holds for every  $t \in \mathbb{T} \upharpoonright S$  that is not minimal in  $\mathbb{T}$ .
- (ii) We say that  $S$  is *non-stationary with respect to*  $\mathbb{T}$  if there is a regressive map  $r : \mathbb{T} \upharpoonright S \rightarrow \mathbb{T}$  with the property that for every  $t \in \mathbb{T}$  there is a function  $c_t : r^{-1}\{t\} \rightarrow \theta_t$  such that  $\theta_t$  is a cardinal smaller than  $\theta$  and  $c_t$  is injective on  $\leq_{\mathbb{T}}$ -chains.
- (iii) The tree  $\mathbb{T}$  is *special* if the set  $\theta$  is non-stationary with respect to  $\mathbb{T}$ .

Todorčević showed that the above definition generalizes the classical definition of special trees of successor height, i.e. trees of height  $\nu^+$  for some infinite cardinal  $\nu$  that are the union of  $\nu$ -many antichains (see [23, Theorem 14]). Moreover, his result shows that a tree of height  $\nu^+$  is special if and only if the set of all ordinals less than  $\nu^+$  of cofinality  $\text{cof}(\nu)$  is non-stationary with respect to the given tree.

**Proposition 1.2.** *If  $S$  is a stationary subset of  $\theta$  that is non-stationary with respect to  $\mathbb{T}$ , then there are no cofinal branches through  $\mathbb{T}$ .*  $\square$

The above proposition directly shows that the non-existence of cofinal branches through special trees is absolute in a strong sense. If  $\mathbb{T}$  is special and  $W$  is an *outer model* of the ground model  $V$  (i.e.  $W$  is a transitive model of ZFC with  $V \subseteq W$  and  $\text{On} \cap V = \text{On} \cap W$ ) with the property that  $\theta$  is a regular cardinal in  $W$ , then there are no cofinal branches through  $\mathbb{T}$  in  $W$ .

In this paper, we want to study special reasons that cause trees without cofinal branches to be non-special in a very absolute way. Examples of such properties were already studied by Baumgartner, Brodsky, Cummings, Devlin, Laver, Rinot, Shelah, Stanley, Todorčević, Torres Pérez and others (see, for example, [4], [6], [7], [22], [25] and [27]). For reasons described later, we will focus on the following property that directly generalizes the concept of cofinal branches and is a consequence of the properties studied in the above papers.

**Definition 1.3.** Let  $\lambda > 0$  be a cardinal. A sequence  $\langle b_\gamma : \lambda \rightarrow \mathbb{T}(\gamma) \mid \gamma < \theta \rangle$  of functions is an *ascending path of width  $\lambda$  through  $\mathbb{T}$*  if for all  $\bar{\gamma} < \gamma < \theta$ , there are  $\alpha, \bar{\alpha} < \lambda$  such that  $b_{\bar{\gamma}}(\bar{\alpha}) <_{\mathbb{T}} b_\gamma(\alpha)$ .

Then the existence of a cofinal branch through  $\mathbb{T}$  is equivalent to the existence of an ascending path of width 1 through  $\mathbb{T}$ . The following lemma shows that the same is true for ascending paths of finite width. The proof of this result is a modification of Baumgartner's elegant proof of [3, Theorem 8.2]. It is contained in Section 3.

**Lemma 1.4.** *If there is an ascending paths of finite width through  $\mathbb{T}$ , then there is a cofinal branch through  $\mathbb{T}$ .*

In combination with the above lemma, the following basic observations show that the notion of an ascending path through a tree of height  $\theta$  is non-trivial if we consider paths of width  $\lambda$  with  $\omega \leq \lambda$  and  $\lambda^+ < \theta$ .

**Proposition 1.5.** (i) *There is an ascending path of width  $\theta$  through  $\mathbb{T}$ .*

- (ii) Assume that  $\theta = \nu^+$  for some cardinal  $\nu$ . Then there is an ascending path of width  $\nu$  through  $\mathbb{T}$  if and only if for every  $\gamma < \theta$  there is  $t \in \mathbb{T}(\gamma)$  such that for every  $\gamma \leq \delta < \theta$  there is  $u \in \mathbb{T}(\delta)$  with  $t <_{\mathbb{T}} u$ .

*Proof.* (i) Fix a sequence  $\langle t_\gamma \in \mathbb{T}(\gamma) \mid \gamma < \theta \rangle$ . Given  $\gamma, \bar{\gamma} < \theta$ , define  $b_\gamma(\bar{\gamma})$  to be the unique element  $s$  of  $\mathbb{T}(\gamma)$  with  $s <_{\mathbb{T}} t_{\bar{\gamma}}$  if  $\gamma < \bar{\gamma}$  and define  $b_\gamma(\bar{\gamma}) = t_\gamma$  otherwise. The resulting sequence  $\langle b_\gamma \mid \gamma < \theta \rangle$  is an ascending path of width  $\theta$  through  $\mathbb{T}$ .

(ii) First, assume that for every  $\gamma < \theta$  there is  $t_\gamma \in \mathbb{T}(\gamma)$  such that for every  $\gamma \leq \delta < \theta$  there is  $u_{\gamma, \delta} \in \mathbb{T}(\delta)$  with  $t_\gamma <_{\mathbb{T}} u_{\gamma, \delta}$ . Given  $\gamma < \theta$ , fix a surjection  $s_\gamma : \nu \rightarrow \gamma + 1$  and define  $b_\gamma(\alpha) = u_{s_\gamma(\alpha), \gamma}$  for all  $\alpha < \nu$ . If  $\bar{\gamma} < \gamma < \theta$ , then there are  $\alpha, \bar{\alpha} < \nu$  with  $\bar{\gamma} = s_\gamma(\alpha) = s_{\bar{\gamma}}(\bar{\alpha})$  and hence  $b_{\bar{\gamma}}(\bar{\alpha}) = t_{\bar{\gamma}} <_{\mathbb{T}} u_{\bar{\gamma}, \gamma} = b_\gamma(\alpha)$ . This shows that the resulting sequence  $\langle b_\gamma \mid \gamma < \theta \rangle$  is an ascending path of width  $\theta$  through  $\mathbb{T}$ .

Now, assume that  $\langle b_\gamma \mid \gamma < \theta \rangle$  is an ascending path of width  $\nu$ . If  $\gamma < \theta$ , then the regularity of  $\theta$  implies that there is  $\alpha < \nu$  with the property that the set  $\{\delta < \theta \mid \exists \beta < \nu b_\gamma(\alpha) <_{\mathbb{T}} b_\delta(\beta)\}$  is unbounded in  $\theta$ . This shows that for all  $\gamma < \theta$  we can find  $\alpha < \nu$  such that for every  $\gamma \leq \delta < \theta$  there is  $u \in \mathbb{T}(\delta)$  with  $b_\gamma(\alpha) <_{\mathbb{T}} u$ .  $\square$

The next lemma shows how ascending paths cause certain trees to be non-special. Its proof is a generalization of the proof of [27, Proposition 2.3]. Given an infinite regular cardinal  $\kappa < \theta$ , we let  $S_\kappa^\theta$  denote the set of all limit ordinals less than  $\theta$  of cofinality  $\kappa$ . Moreover, given some cardinal  $\lambda < \theta$ , we let  $S_{>\lambda}^\theta$  denote the set of all limit ordinals less than  $\theta$  of cofinality greater than  $\lambda$ . The sets  $S_{<\lambda}^\theta$ ,  $S_{\leq \kappa}^\theta$  and  $S_{\geq \kappa}^\theta$  are defined analogously

**Lemma 1.6.** *Let  $\lambda < \theta$  be a cardinal with the property that  $\theta$  is not a successor of a cardinal of cofinality less than or equal to  $\lambda$  and let  $S \subseteq S_{>\lambda}^\theta$  be stationary in  $\theta$ . If  $S$  is non-stationary with respect to  $\mathbb{T}$ , then there is no ascending path of width  $\lambda$  through  $\mathbb{T}$ .*

*Proof.* Assume, towards a contradiction, that  $\langle b_\gamma : \lambda \rightarrow \mathbb{T}(\gamma) \mid \gamma < \theta \rangle$  is an ascending path through  $\mathbb{T}$ . Let  $r : \mathbb{T} \upharpoonright S \rightarrow \mathbb{T}$  and  $\langle c_t : r^{-1}\{t\} \rightarrow \theta_t \mid t \in \mathbb{T} \rangle$  witness that  $S$  is non-stationary with respect to  $\mathbb{T}$ . Then there is a club  $C$  in  $\theta$  with  $\theta_{r(b_{\bar{\gamma}}(\alpha))} < \gamma$  for all  $\gamma \in C$ ,  $\bar{\gamma} < \gamma$  and  $\alpha < \lambda$ . Since  $S \subseteq S_{>\lambda}^\theta$  is stationary in  $\theta$ , we can find  $\delta < \theta$  and  $E \subseteq C \cap S$  stationary in  $\theta$  such that the following statements hold for all  $\delta \in E$  and  $\alpha < \lambda$ :

- (i)  $r(b_\gamma(\alpha)) \in \mathbb{T}_{<\delta}$ .
- (ii) If  $r(b_\gamma(\alpha)) = r(b_{\bar{\gamma}}(\alpha))$  for some  $\bar{\gamma} < \gamma$ , then  $c_{r(b_\gamma(\alpha))}(b_\gamma(\alpha)) < \delta$ .

By our assumptions, there is a cardinal  $\nu < \theta$  of cofinality greater than  $\lambda$  and a surjection  $s : \nu \rightarrow \delta$ . Fix a strictly increasing cofinal sequence  $\langle \nu_\xi \mid \xi < \text{cof}(\nu) \rangle$  in  $\nu$ . Given  $\gamma \in E$ , there is a minimal  $\xi_\gamma < \text{cof}(\nu)$  such that for all  $\alpha < \lambda$ , there are  $\zeta_0, \zeta_1 < \nu_{\xi_\gamma}$  with  $r(b_\gamma(\alpha)) \in \mathbb{T}(s(\zeta_0))$  and, if  $r(b_\gamma(\alpha)) = r(b_{\bar{\gamma}}(\alpha))$  for some  $\bar{\gamma} < \gamma$ , then  $c_{r(b_\gamma(\alpha))}(b_\gamma(\alpha)) = s(\zeta_1)$ . Then there is  $U \subseteq E$  unbounded in  $\theta$  and  $\xi_* < \text{cof}(\nu)$  such that  $\xi_* = \xi_\gamma$  for all  $\gamma$  in  $U$ .

Let  $\gamma_*$  denote that  $\nu$ -th element in the monotone enumeration of  $U$ . Given  $\gamma \in U \cap \gamma_*$ , our assumptions imply that there are  $\alpha, \beta < \lambda$  with  $b_\gamma(\alpha) <_{\mathbb{T}} b_{\gamma_*}(\beta)$ . Since the cofinality of  $\nu$  is greater than  $\lambda$ , there are  $\alpha_*, \beta_* < \lambda$  such that the set

$$A = \{\gamma \in U \cap \gamma_* \mid b_\gamma(\alpha_*) <_{\mathbb{T}} b_{\gamma_*}(\beta_*)\}$$

has cardinality  $\nu$ . Pick a function  $f : A \rightarrow \nu_{\xi_*} \times \nu_{\xi_*}$  with the property that  $f(\gamma) = \langle \zeta_0, \zeta_1 \rangle$  implies that  $r(b_\gamma(\alpha_*)) \in \mathbb{T}(s(\zeta_0))$  and, if  $r(b_\gamma(\alpha_*)) = r(b_{\bar{\gamma}}(\alpha_*))$  for some  $\bar{\gamma} < \gamma$ , then  $c_{r(b_\gamma(\alpha_*))}(b_\gamma(\alpha_*)) = s(\zeta_1)$ . By our assumptions, there are  $\gamma_0, \gamma_1, \gamma_2 \in A$  and  $\zeta_0, \zeta_1 < \nu_{\xi_*}$  such that  $\gamma_0 < \gamma_1 < \gamma_2$  and  $f(\gamma_i) = \langle \zeta_0, \zeta_1 \rangle$  for all  $i < 3$ . Given  $i < 3$ , we have  $r(b_{\gamma_i}(\alpha_*)) \in \mathbb{T}(s(\zeta_0))$  and

$$r(b_{\gamma_i}(\alpha_*)) <_{\mathbb{T}} b_{\gamma_i}(\alpha_*) <_{\mathbb{T}} b_{\gamma_*}(\beta_*).$$

This implies that  $r(b_{\gamma_0}(\alpha_*)) = r(b_{\gamma_1}(\alpha_*)) = r(b_{\gamma_2}(\alpha_*))$ . In this situation, the above choices ensure that

$$c_{r(b_{\gamma_0}(\alpha_*))}(b_{\gamma_1}(\alpha_*)) = s(\zeta_1) = c_{r(b_{\gamma_0}(\alpha_*))}(b_{\gamma_2}(\alpha_*))$$

holds. This implies that the nodes  $b_{\gamma_1}(\alpha_*)$  and  $b_{\gamma_2}(\alpha_*)$  are incompatible in  $\mathbb{T}$ , a contradiction.  $\square$

**Corollary 1.7.** *Let  $\lambda < \theta$  be a cardinal with the property that  $\theta$  is not a successor of a cardinal of cofinality less than or equal to  $\lambda$ . If  $\mathbb{T}$  contains an ascending path of width  $\lambda$ , then  $\mathbb{T}$  is not special.*

*Proof.* Our assumptions imply that the set  $S_{\lambda^+}^\theta$  is stationary in  $\theta$  and non-stationary with respect to  $\mathbb{T}$ . In this situation, the statement of the corollary follows directly from Lemma 1.6.  $\square$

Note that the above result leaves open the question whether there can be special trees whose height is the successor of a singular cardinal  $\nu$  that contain an ascending path of width  $\lambda$  with  $\text{cof}(\nu) \leq \lambda < \nu$  (see Question 6.1).

The above corollary shows that ascending paths cause trees to be non-special in an absolute way: in the situation of the corollary, the tree  $\mathbb{T}$  remains non-special in every outer model in which  $\theta$  and  $\lambda$  satisfy the assumptions of the corollary. We will later show that, if  $\theta$  and  $\lambda$  satisfy certain cardinal arithmetic assumptions, then the converse of this implication also holds true, i.e. if there is no ascending path of width  $\lambda$  through  $\mathbb{T}$ , then  $\mathbb{T}$  is special in a forcing extension of the ground model in which the above assumptions on  $\theta$  and  $\lambda$  hold. This follows from the fact that ascending paths are closely related to maximal antichains in the canonical partial order that specializes a tree of uncountable regular height.

**Definition 1.8.** Let  $\kappa < \theta$  be an infinite regular cardinal. We define  $\mathbb{P}_\kappa(\mathbb{T})$  to be the partial order that consists of partial functions from  $\mathbb{T}$  to  $\kappa$  of cardinality less than  $\kappa$  that are injective on chains in  $\mathbb{T}$  and are ordered by reversed inclusion.

It is easy to see that partial orders of the form  $\mathbb{P}_\kappa(\mathbb{T})$  are  $< \kappa$ -closed and forcing with  $\mathbb{P}_\kappa(\mathbb{T})$  collapses every cardinal in the interval  $(\kappa, \theta)$ . Moreover, if forcing with  $\mathbb{P}_\kappa(\mathbb{T})$  preserves the regularity of  $\theta$ , then the tree  $\mathbb{T}$  is special in all  $\mathbb{P}_\kappa(\mathbb{T})$ -generic extensions. Therefore it is natural to ask under which conditions this regularity is preserved. We will later show (see Corollary 2.2) that the assumption that forcing with  $\mathbb{P}_\kappa(\mathbb{T})$  preserves the regularity of  $\theta$  implies that  $\mu^{< \kappa} < \theta$  holds for all  $\mu < \theta$ . The following result shows that under this cardinal arithmetic assumption, we can characterize this preservation by the non-existence of ascending paths of small width.

**Theorem 1.9.** *The following statements are equivalent for every infinite regular cardinal  $\kappa < \theta$  with  $\mu^{< \kappa} < \theta$  for all  $\mu < \theta$ :*

- (i) *There is no ascending path of width less than  $\kappa$  through  $\mathbb{T}$ .*

- (ii) *The partial order  $\mathbb{P}_\kappa(\mathbb{T})$  satisfies the  $\theta$ -chain condition.*
- (iii) *Forcing with the partial order  $\mathbb{P}_\kappa(\mathbb{T})$  preserves the regularity of  $\theta$ .*

By the above remarks, in the setting of Theorem 1.9, the three statements listed in the theorem are also equivalent to the statement that there is some outer model  $W$  of the ground model  $V$  such that  $\kappa$  and  $\theta$  are regular cardinals in  $W$  and  $\mathbb{T}$  is a special tree in  $W$ . Given an uncountable regular cardinal  $\kappa$  with  $\kappa = \kappa^{<\kappa}$ , the above theorem allows us to show that the collection of *specializable* trees of height  $\kappa^+$  (i.e. the collection of all trees that are special in a cofinality-preserving outer model of the ground model  $V$ ) can be defined through the existence of ascent paths of width less than  $\kappa$ . It is not known to the author whether this collection is also definable if  $\kappa < \kappa^{<\kappa}$  (see Question 6.4).

In combination with Lemma 1.4, the above theorem directly implies the statement of a classical result of Baumgartner (see [3, Theorem 8.2] and [5, Lemma 5.3]) stating that partial order  $\mathbb{P}_\omega(\mathbb{T})$  satisfies the  $\theta$ -chain condition if and only if there is no cofinal branch through  $\mathbb{T}$ .

We present an application of the above result to questions regarding potential generalizations of Martin's Axiom to larger cardinalities. Given a partial order  $\mathbb{P}$ , we let  $\text{FA}_\theta(\mathbb{P})$  denote the statement that for every collection  $\mathcal{D}$  of  $\theta$ -many dense subsets of  $\mathbb{P}$ , there is a  $\mathcal{D}$ -generic filter, i.e. a filter  $F$  on  $\mathbb{P}$  with  $D \cap F \neq \emptyset$  for all  $D \in \mathcal{D}$ . A result of Shelah (see [21, Theorem 6]) shows that CH implies that there is a  $\sigma$ -closed partial order  $\mathbb{P}$  satisfying the  $\aleph_2$ -chain condition such that  $\text{FA}_{\aleph_2}(\mathbb{P})$  fails. This partial order is not *well-met*, i.e. there are compatible conditions without a greatest lower bound in this partial order. Since recent work of Shelah (see [20]) shows that some well-met condition is necessary for such generalizations of Martin's Axiom to hold and results of Baumgartner and Shelah (see [3, Section 4] and [18]) show that such forcing axioms can consistently hold for all  $\sigma$ -closed, well-met partial order satisfying certain strengthenings of the  $\aleph_2$ -chain condition, it is natural to ask whether the statement that  $\text{FA}_{\aleph_2}(\mathbb{P})$  holds for all  $\sigma$ -closed, well-met partial orders  $\mathbb{P}$  satisfying the  $\aleph_2$ -chain conditions is consistent. With the help of Todorćević's method of *walks on ordinals* and a result of Todorćević from [25], we will prove the following result that shows that the consistency strength of such a forcing axiom is at least a weakly compact cardinal.

**Theorem 1.10.** *Let  $\kappa$  be an uncountable regular cardinal with  $\kappa = \kappa^{<\kappa}$ . If  $\kappa^+$  is not weakly compact in  $L$ , then there is a  $<\kappa$ -closed, well-met partial order  $\mathbb{P}$  satisfying the  $\kappa^+$ -chain condition with the property that  $\text{FA}_{\kappa^+}(\mathbb{P})$  fails.*

Next, we discuss an application of the notion of ascending paths to questions about the productivity of certain chain conditions and characterizations of weak compactness (see, for example, [5] and [16]). Baumgartner's result mentioned above shows that, if  $\mathbb{T}$  has no cofinal branches, then the partial order  $\mathbb{P}_\omega(\mathbb{T})$  satisfies the  $\theta$ -chain condition. Since special trees have no cofinal branches, this argument actually shows that finite support products of the partial order  $\mathbb{P}_\omega(\mathbb{T})$  satisfy the  $\theta$ -chain condition in this case. The next theorem is a strengthening of Theorem 1.9. It shows that ascending paths of infinite width provide examples of trees where this chain condition fails in infinite products.

**Theorem 1.11.** *The following statements are equivalent for every infinite regular cardinal  $\kappa < \theta$  with  $\mu^{<\kappa} < \theta$  for all  $\mu < \theta$ .*

- (i) *There is no ascending path of width less than  $\kappa$  through  $\mathbb{T}$ .*

- (ii) If  $\nu \leq \kappa$  is an infinite regular cardinal, then  $<\kappa$ -support products of the partial order  $\mathbb{P}_\nu(\mathbb{T})$  satisfy the  $\theta$ -chain condition.

In the following, we are interested in the infinite productivity of stronger chain conditions. Remember that a partial order  $\mathbb{P}$  is  $\theta$ -Knaster if every set of  $\theta$ -many conditions in  $\mathbb{P}$  contains a subset of cardinality  $\theta$  consisting of pairwise compatible conditions. This property of partial orders is of great interest, because it implies the  $\theta$ -chain condition and is preserved under finite products. Moreover, it is easy to see that, if  $\theta$  is weakly compact, then the class of  $\theta$ -Knaster partial orders is closed under  $<\nu$ -support products for every  $\nu < \theta$  (see [5, Proposition 1.1] and the proof of Theorem 1.12 in Section 4). It is now natural to ask whether this productivity characterizes weakly compact cardinals. It was shown by Cox and the author that it is consistent that there is an inaccessible cardinal  $\vartheta$  that is not weakly compact and has the property that the class of  $\vartheta$ -Knaster partial orders is closed under  $\nu$ -support products for all  $\nu < \vartheta$  (see [5, Theorem 1.14]). In contrast, we will use Theorem 1.11 to show that the infinite productivity of the  $\vartheta$ -Knaster property characterizes weak compactness in canonical inner models of set theory (so-called *Jensen-style extender models*, see [29]). The proof of the following theorem relies on Todorćević's method of walks on ordinals and results of Schimmerling and Zeman on the existence of square sequences in canonical inner models (see [17] and [30]) that extend seminal results of Jensen from [8].

**Theorem 1.12.** *Let  $L[E]$  be a Jensen-style extender model. In  $L[E]$ , the following statements are equivalent for every uncountable regular cardinal  $\vartheta$ :*

- (i)  $\vartheta$  is weakly compact.
- (ii) The class of  $\vartheta$ -Knaster partial orders is closed under  $\nu$ -support products for all  $\nu < \vartheta$ .
- (iii)  $\vartheta$  is not the successor of a subcompact cardinal and the class of  $\vartheta$ -Knaster partial orders is closed under countable support products.

In particular, it is consistent with the axioms of ZFC that weak compactness is characterized by the countable productivity of the Knaster property.

Next, we present result concerning the existence and non-existence of trees without cofinal branches containing ascending paths of small width. The proofs of most these results make use of the notion of *narrow system* introduced by Magidor and Shelah in [15] and recent results of Lambie-Hanson about these systems contained in [12]. The statements (i), (ii) and (v) of the following theorem are direct consequences of results contained in [12]. Moreover, the statement (iii) is implicitly proven in the base case of the inductive proof of the main theorem of [14]. Remember that a regular cardinal  $\kappa$  is *indestructibly weakly compact* if  $\kappa$  is weakly compact in every forcing extension by a  $<\kappa$ -closed partial order.

**Theorem 1.13.** (i) *If  $\theta$  is weakly compact, then every tree of height  $\theta$  that contains an ascending path of width less than  $\theta$  has a cofinal branch.*

- (ii) *If  $\kappa \leq \theta$  is a  $\theta$ -compact cardinal, then every tree of height  $\theta$  that contains an ascending path of width less than  $\kappa$  has a cofinal branch.*
- (iii) *If  $\theta$  is weakly compact,  $\kappa < \theta$  is an uncountable regular cardinal and  $G$  is  $\text{Col}(\kappa, <\theta)$ -generic over  $V$ , then in  $V[G]$  every tree of height  $\theta$  that contains an ascending path of width less than  $\kappa$  has a cofinal branch.*
- (iv) *If  $\kappa \leq \theta$  is indestructibly weakly compact, then every tree of height  $\theta$  that contains an ascending path of width less than  $\kappa$  has a cofinal branch.*

- (v) If  $\kappa \leq \theta$  is  $\theta$ -compact,  $\nu < \kappa$  is an uncountable regular cardinal and  $G$  is  $\text{Col}(\nu, < \kappa)$ -generic over  $V$ , then in  $V[G]$  every tree of height  $\theta$  that contains an ascending path of width less than  $\nu$  has a cofinal branch.

Moreover, a result of Lambie-Hanson shows that it is possible to use very strong large cardinal assumptions to prove global non-existence results. In the model constructed in the proof of [12, Theorem 5.2], every tree of uncountable regular height  $\vartheta$  containing an ascending path of width  $\lambda$  with  $\lambda^+ < \vartheta$  has a cofinal branch.

In contrast to the above non-existence results, we will also show that trees without cofinal branches containing ascending paths of small width can be constructed from certain combinatorial principles (see Theorem 4.12 and Theorem 5.8). These constructions allow us to derive lower bounds for some of the consistency results listed in Theorem 1.13.

Finally, we study the influence of the *Proper Forcing Axiom* PFA on trees containing ascending paths. We show that PFA implies an analogue of Lemma 1.4 for trees of height greater than  $\omega_1$  that contain ascending path of countable width. In contrast, we show that PFA does not prove a similar conclusion for ascending paths of width  $\omega_1$ . Remember that  $\mathbb{T}$  is a  $\theta$ -Souslin tree if the partial order induced by  $\mathbb{T}$  satisfies the  $\theta$ -chain condition. The proof of the second statement of the next theorem relies on a construction of an  $\omega_3$ -Souslin tree containing an ascending path of width  $\omega_1$  using a partial square principle introduced by Baumgartner (see Definition 5.5).

**Theorem 1.14.** *Assume that PFA holds.*

- (i) *If  $\theta > \omega_1$ , then every tree of height  $\theta$  that contains an ascending path of width  $\omega$  has a cofinal branch.*
- (ii) *There is a partial order  $\mathbb{P}$  with the property that, whenever  $G$  is  $\mathbb{P}$ -generic over  $V$ , then PFA holds in  $V[G]$  and in  $V[G]$  there is an  $\omega_3$ -Souslin tree that contains an ascending path of width  $\omega_1$ .*
- (iii) *If  $\kappa$  is a strongly compact cardinal and  $G$  is  $\text{Col}(\omega_2, < \kappa)$ -generic over  $V$ , then, in  $V[G]$ , PFA holds and every tree of regular height greater than  $\omega_2$  that contains an ascending path of width  $\omega_1$  has a cofinal branch.*

We outline the structure of this paper. Section 2 contains the proofs of Theorem 1.9 and Theorem 1.11. In Section 3, we will prove Lemma 1.4 and Theorem 1.13. The proofs of Theorem 1.10 and Theorem 1.12 are contained in Section 4. Theorem 1.14 is proven in Section 5. In Section 6, we list several open questions motivated by the above results.

## 2. ASCENDING PATHS AND ANTICHAINS

This section is devoted to the proofs of Theorem 1.9 and Theorem 1.11. We start by showing that the cardinal arithmetic assumptions of Theorem 1.9 are necessary for the equivalence of the statements listed in the theorem.

Given a regular cardinal  $\kappa < \theta$ , let  $\text{Add}(\kappa, \theta)$  denote the partial order consisting of partial functions  $f : \theta \xrightarrow{\text{par}} 2$  of cardinality less than  $\kappa$  ordered by reversed inclusion. Then forcing with  $\text{Add}(\kappa, \theta)$  adds  $\theta$ -many Cohen subsets of  $\kappa$  to the ground model. Moreover, given an ordinal  $\mu \geq \kappa$ , we let  $\text{Col}(\kappa, \mu)$  denote the partial order consisting of partial injections  $i : \kappa \xrightarrow{\text{par}} \mu$  of cardinality less than  $\kappa$  ordered by reversed inclusion. This partial order is forcing equivalent to the usual Levy collapse of  $\mu$  to  $\kappa$ .

**Proposition 2.1.** *If  $\kappa < \theta$  is a regular cardinal and  $\kappa \leq \mu < \theta$ , then there is a forcing projection  $\pi : \mathbb{P}_\kappa(\mathbb{T}) \longrightarrow \text{Add}(\kappa, \theta) \times \text{Col}(\kappa, \mu)$ .*

*Proof.* Fix a sequence  $\langle t_\gamma \in \mathbb{T}(\gamma) \mid \gamma < \theta \rangle$  with  $t_\gamma <_{\mathbb{T}} t_\mu$  for all  $\gamma < \mu$ . Pick a condition  $p$  in  $\mathbb{P}_\kappa(\mathbb{T})$ . Define  $f(p)$  to be the unique condition in  $\text{Add}(\kappa, \theta)$  with  $\text{dom}(f(p)) = \{\gamma < \theta \mid t_{\mu+\gamma} \in \text{dom}(p)\}$  and

$$f(p)(\gamma) = 1 \iff p(t_{\mu+\gamma}) \text{ is an odd ordinal}$$

for all  $\gamma \in \text{dom}(f(p))$ . Moreover define  $i(p)$  to be the unique condition in  $\text{Col}(\kappa, \mu)$  with  $\text{dom}(i(p)) = \{p(t_\gamma) \mid \gamma < \mu, t_\gamma \in \text{dom}(p)\}$  and  $i(p)(p(t_\gamma)) = \gamma$  for all  $\gamma < \mu$  with  $t_\gamma \in \text{dom}(p)$ . It is easy to check that the resulting map

$$\pi : \mathbb{P}_\kappa(\mathbb{T}) \longrightarrow \text{Add}(\kappa, \theta) \times \text{Col}(\kappa, \mu); p \longmapsto \langle f(p), i(p) \rangle$$

is a forcing projection.  $\square$

**Corollary 2.2.** *If  $\kappa < \theta$  is a regular cardinal such that forcing with  $\mathbb{P}_\kappa(\mathbb{T})$  preserves the regularity of  $\theta$ , then  $\mu^{<\kappa} < \theta$  holds for all  $\mu < \theta$ .*

*Proof.* Assume, towards a contradiction, that there is  $\kappa \leq \mu < \theta$  with  $\mu^{<\kappa} \geq \theta$ . Let  $G$  be  $\mathbb{P}_\kappa(\mathbb{T})$ -generic over  $V$ . By Proposition 2.1, we can find  $H_0, H_1 \in V[G]$  such that  $H_0$  is  $\text{Add}(\kappa, \theta)$  over  $V$  and  $H_1$  is  $\text{Col}(\kappa, \mu)$ -generic over  $V$ . Then  $V[H_0]$  contains a bijection between  $\kappa$  and  $\kappa^{<\kappa}$  and  $V[H_1]$  contains a bijection between  $\kappa$  and  $\mu$ . This shows that  $V[G]$  contains a surjection from  $\kappa$  onto  $\theta$ .  $\square$

The following proposition shows how ascending paths induce antichains in infinite products of partial orders of the form  $\mathbb{P}_\kappa(\mathbb{T})$ .

**Proposition 2.3.** *If  $\langle b_\gamma : \lambda \longrightarrow \mathbb{T}(\gamma) \mid \gamma < \theta \rangle$  is an ascending path through  $\mathbb{T}$  and  $\kappa$  is an infinite regular cardinal, then the full support product  $\prod_{\lambda \times \lambda} \mathbb{P}_\kappa(\mathbb{T})$  does not satisfy the  $\theta$ -chain condition.*

*Proof.* Given  $\gamma < \theta$ , let  $\vec{p}_\gamma$  denote the unique condition in  $\prod_{\lambda \times \lambda} \mathbb{P}_\kappa(\mathbb{T})$  with the property that  $\text{dom}(\vec{p}_\gamma(\alpha, \bar{\alpha})) = \{b_\gamma(\alpha), b_\gamma(\bar{\alpha})\}$  and

$$\vec{p}_\gamma(\alpha, \bar{\alpha})(b_\gamma(\alpha)) = \vec{p}_\gamma(\alpha, \bar{\alpha})(b_\gamma(\bar{\alpha})) = 0$$

for all  $\alpha, \bar{\alpha} < \lambda$ . By our assumption, the sequence  $\langle \vec{p}_\gamma \mid \gamma < \theta \rangle$  is an injective enumeration of an antichain in  $\prod_{\lambda \times \lambda} \mathbb{P}_\kappa(\mathbb{T})$ .  $\square$

The starting point of the proof of Theorem 1.11 is the following basic observation.

**Proposition 2.4.** *If  $\kappa < \theta$  is an infinite regular cardinal and  $p, q \in \mathbb{P}_\kappa(\mathbb{T})$  are incompatible, then either  $p \upharpoonright (\text{dom}(p) \cap \text{dom}(q)) \neq q \upharpoonright (\text{dom}(p) \cap \text{dom}(q))$  or there are  $s \in \text{dom}(p) \setminus \text{dom}(q)$  and  $t \in \text{dom}(q) \setminus \text{dom}(p)$  such that  $p(s) = q(t)$  and the nodes  $s$  and  $t$  are compatible in  $\mathbb{T}$ .  $\square$*

In particular, given regular cardinals  $\nu \leq \kappa < \theta$ , every antichain in  $\mathbb{P}_\nu(\mathbb{T})$  is an antichain in  $\mathbb{P}_\kappa(\mathbb{T})$ .

**Proposition 2.5.** *Let  $\kappa$  be an infinite regular cardinal and let  $\mathbb{D}_\kappa(\mathbb{T})$  denote the set of all conditions  $p$  in  $\mathbb{P}_\kappa(\mathbb{T})$  with the property that for all  $s, u \in \text{dom}(p)$  with  $\text{lh}_{\mathbb{T}}(s) < \text{lh}_{\mathbb{T}}(u)$ , there is  $t \in \text{dom}(p)$  with  $\text{lh}_{\mathbb{T}}(s) = \text{lh}_{\mathbb{T}}(t)$  and  $t <_{\mathbb{T}} u$ . Then the set  $\mathbb{D}_\kappa(\mathbb{T})$  is dense in  $\mathbb{P}_\kappa(\mathbb{T})$ .*

*Proof.* Pick a condition  $p$  in  $\mathbb{P}_\kappa(\mathbb{T})$  and set  $A = \{\text{lh}_\mathbb{T}(t) \mid t \in \text{dom}(p)\}$ . Define  $D$  to be the set of all  $s \in \mathbb{T}$  such that  $\text{lh}_\mathbb{T}(s) \in A$  and  $s \leq_\mathbb{T} t$  for some  $t \in \text{dom}(p)$ . Then  $D$  is a subset of  $\mathbb{T}$  of cardinality less than  $\kappa$  with  $\text{dom}(p) \subseteq D$  and we can find a function  $q : D \rightarrow \kappa$  such that  $q \upharpoonright \text{dom}(p) = p$ ,  $q \upharpoonright (D \setminus \text{dom}(p))$  is an injection and  $q[D \setminus \text{dom}(p)] \subseteq \kappa \setminus \text{ran}(p)$ . We can conclude that  $q \in \mathbb{D}_\kappa(\mathbb{T})$  with  $q \leq_{\mathbb{P}_\kappa(\mathbb{T})} p$ .  $\square$

In the proof of Theorem 1.11, we want to restrict ourselves to trees that satisfy the following normality condition.

**Definition 2.6.** We say that the tree  $\mathbb{T}$  *does not split at limit levels* if for all  $\gamma \in \theta \cap \text{Lim}$  and all  $t_0, t_1 \in \mathbb{T}(\gamma)$  with  $t_0 \neq t_1$ , we can find  $\bar{\gamma} < \gamma$  and  $s_0, s_1 \in \mathbb{T}(\bar{\gamma})$  such that  $s_0 \neq s_1$  and  $s_i <_\mathbb{T} t_i$  for all  $i < 2$ .

Note that a standard construction (see [10, Section III.3]) shows that for every tree  $\mathbb{T}$  of height  $\theta$  there is a tree  $\bar{\mathbb{T}}$  of height  $\theta$  that does not split at limit levels such that  $\mathbb{T}$  is isomorphic to the tree  $\bar{\mathbb{T}} \upharpoonright (\theta \setminus \text{Lim})$ . Note that this means that the existence of an ascending path of width  $\lambda$  through  $\bar{\mathbb{T}}$  implies the existence of an ascending path of width  $\lambda$  through  $\mathbb{T}$ . Moreover, Proposition 2.4 shows that every antichain in a product of the partial order  $\mathbb{P}_\kappa(\mathbb{T})$  induces an antichain of the same size in the corresponding product of the partial order  $\mathbb{P}_\kappa(\bar{\mathbb{T}})$ . In combination, this shows that, in order to prove the implication from (i) to (ii) in Theorem 1.11, it suffices to prove this implication for all trees of height  $\theta$  that do not split at limit levels.

*Proof of Theorem 1.11.* Let  $\nu \leq \kappa < \theta$  be infinite regular cardinals with  $\mu^{<\kappa} < \theta$  for all  $\mu < \theta$ .

First, assume that  $\mathbb{T}$  contains an ascending path of width  $\lambda < \kappa$ . Then Proposition 2.3 shows that for every regular cardinal  $\nu \leq \kappa$ , the full support product  $\prod_{\lambda \times \lambda} \mathbb{P}_\nu(\mathbb{T})$  does not satisfy the  $\theta$ -chain condition.

In the other direction, assume that there is a  $<\kappa$ -support product of the partial order  $\mathbb{P}_\nu(\mathbb{T})$  that does not satisfy the  $\theta$ -chain condition. By the above remarks, we may assume that  $\mathbb{T}$  does not split at limit levels. Moreover, our cardinal arithmetic assumption allows us to find  $\mu < \kappa$  and an injective enumeration  $\langle \bar{p}_\gamma \mid \gamma < \theta \rangle$  of an antichain in the full support product  $\mathbb{P} = \prod_\mu \mathbb{P}_\nu(\mathbb{T})$ . By Proposition 2.5, there is a sequence  $\langle \bar{q}_\gamma \mid \gamma < \theta \rangle$  of conditions in  $\mathbb{P}$  such that  $\bar{q}_\gamma \leq_{\mathbb{P}} \bar{p}_\gamma$  and  $\bar{q}_\gamma(\beta) \in \mathbb{D}_\nu(\mathbb{T})$  for all  $\gamma < \theta$  and  $\beta < \mu$ .

Fix  $\gamma \in S_\kappa^\theta$ . Set  $A_\gamma = \{\langle t, \beta \rangle \in \mathbb{T} \times \mu \mid t \in \text{dom}(\bar{q}_\gamma(\beta))\}$  and let  $\langle t_\gamma(\alpha) \mid \alpha < \lambda_\gamma \rangle$  be a bijective enumeration of the set

$$\{t \in \mathbb{T}(\gamma) \mid \exists \beta < \mu \exists u \in \text{dom}(\bar{q}_\gamma(\beta)) t \leq_\mathbb{T} u\}$$

for a cardinal  $\lambda_\gamma < \kappa$ . Since  $\mathbb{T}$  does not split at limit levels, there is  $r(\gamma) < \gamma$  and an injection  $\iota_\gamma : \lambda_\gamma \rightarrow \mathbb{T}(r(\gamma))$  such that  $A_\gamma \cap (\mathbb{T}_{<\gamma} \times \mu) \subseteq \mathbb{T}_{<r(\gamma)} \times \mu$  and  $\iota_\gamma(\alpha) <_\mathbb{T} t_\gamma(\alpha)$  for all  $\alpha < \lambda_\gamma$ .

By our assumptions, we may apply Fodor's Lemma to find  $\lambda < \kappa$ ,  $\rho < \theta$ ,  $E \subseteq S_\kappa^\theta$  stationary in  $\theta$  and a sequence  $\langle H_\beta \subseteq \rho \mid \beta < \mu \rangle$  such that  $\lambda = \lambda_\gamma$ ,  $\rho = r(\gamma)$  and  $H_\beta = \{\text{lh}_\mathbb{T}(s) \mid s \in \text{dom}(\bar{q}_\gamma(\beta)) \cap \mathbb{T}_{<\gamma}\}$  for all  $\gamma \in E$  and  $\beta < \mu$ . In this situation, our cardinal arithmetic assumption allows us to use the  $\Delta$ -system lemma to find  $F \subseteq E$  unbounded in  $\theta$ ,  $Q \subseteq \mathbb{T} \times \mu$  and  $R \subseteq \mathbb{T}(\rho)$  such that the set  $\{A_\gamma \mid \gamma \in F\}$  is a  $\Delta$ -system with root  $Q$  and the set  $\{\text{ran}(\iota_\gamma) \mid \gamma \in F\}$  is a  $\Delta$ -system with root  $R$ . Next, we use the pigeonhole principle and our cardinal arithmetic assumption to find  $U \subseteq F$  unbounded in  $\theta$ ,  $B \subseteq \lambda$ , a map  $c : Q \rightarrow \nu$  and an injection  $\iota : B \rightarrow R$

such that  $c(t, \beta) = \vec{q}_\gamma(\beta)(t)$ ,  $B = \{\alpha < \lambda \mid \iota_\gamma(\alpha) \in R\}$  and  $\iota(\alpha) = \iota_\gamma(\alpha)$  for all  $\gamma \in U$ ,  $\alpha \in B$  and  $\langle t, \beta \rangle \in Q$ .

Pick  $\gamma_0, \gamma_1 \in U$  such that  $\gamma_0 < \gamma_1$  and  $A_{\gamma_0} \subseteq \mathbb{T}_{<\gamma_1} \times \mu$ . Since the conditions  $\vec{q}_{\gamma_0}$  and  $\vec{q}_{\gamma_1}$  are incompatible in  $\mathbb{P}$ , there is a  $\beta < \mu$  such that the conditions  $\vec{q}_{\gamma_0}(\beta)$  and  $\vec{q}_{\gamma_1}(\beta)$  are incompatible in  $\mathbb{P}_\nu(\mathbb{T})$ . By the above choices, we have  $\vec{q}_{\gamma_0}(\beta)(t) = c(t, \beta) = \vec{q}_{\gamma_1}(\beta)$  for all  $t \in \text{dom}(\vec{q}_{\gamma_0}(\beta)) \cap \text{dom}(\vec{q}_{\gamma_1}(\beta))$ . In this situation, Proposition 2.4 shows that there are  $t_0 \in \text{dom}(\vec{q}_{\gamma_0}(\beta))$  and  $t_1 \in \text{dom}(\vec{q}_{\gamma_1}(\beta))$  such that  $\vec{q}_{\gamma_0}(\beta)(t_0) = \vec{q}_{\gamma_1}(\beta)(t_1)$  and  $t_I <_{\mathbb{T}} t_{1-I}$  for some  $I < 2$ .

Assume, towards a contradiction, that  $t_I \in \mathbb{T}_{<\gamma_I}$ . Then  $\text{lh}_{\mathbb{T}}(t_I) \in H_\beta$  and there is an  $s \in \text{dom}(\vec{q}_{\gamma_{1-I}}(\beta))$  with  $\text{lh}_{\mathbb{T}}(s) = \text{lh}_{\mathbb{T}}(t_I)$ . Since  $\vec{q}_{\gamma_{1-I}}(\beta) \in \mathbb{D}_\nu(\mathbb{T})$ , we can find  $t \in \text{dom}(\vec{q}_{\gamma_{1-I}}(\beta))$  with  $t <_{\mathbb{T}} t_{1-I}$  and  $\text{lh}_{\mathbb{T}}(s) = \text{lh}_{\mathbb{T}}(t)$ . But then  $t = t_I \in Q$  and

$$\vec{q}_{\gamma_{1-I}}(\beta)(t_I) = c(t_I, \beta) = \vec{q}_{\gamma_I}(\beta)(t_I) = \vec{q}_{\gamma_{1-I}}(\beta)(t_{1-I}),$$

a contradiction.

The above computations show that  $t_I \notin \mathbb{T}_{<\gamma_I}$  and this implies that  $I = 0$  and  $t_1 \notin \mathbb{T}_{<\gamma_1}$ , because  $\text{dom}(\vec{q}_{\gamma_0}(\beta)) \subseteq \mathbb{T}_{<\gamma_1}$  and  $\text{dom}(\vec{q}_{\gamma_1}(\beta)) \cap \mathbb{T}_{<\gamma_1} \subseteq \mathbb{T}_{<\rho}$ . Then there are  $\alpha_0, \alpha_1 < \lambda$  with  $t_{\gamma_0}(\alpha_0) \leq_{\mathbb{T}} t_0$  and  $t_{\gamma_1}(\alpha_1) \leq_{\mathbb{T}} t_1$ . In particular, this implies that  $\lambda > 0$ . By the above choices, we have  $\iota_{\gamma_0}(\alpha_0) <_{\mathbb{T}} t_{\gamma_0}(\alpha_0) \leq_{\mathbb{T}} t_0 <_{\mathbb{T}} t_1$  and  $\iota_{\gamma_1}(\alpha_1) <_{\mathbb{T}} t_{\gamma_1}(\alpha_1) \leq_{\mathbb{T}} t_1$ . This implies that  $\iota_{\gamma_0}(\alpha_0) = \iota_{\gamma_1}(\alpha_1) \in R$  and  $\alpha_0, \alpha_1 \in B$ . But then  $\iota(\alpha_0) = \iota_{\gamma_0}(\alpha_0) = \iota_{\gamma_1}(\alpha_1) = \iota(\alpha_1)$  and  $\alpha_0 = \alpha_1$ . Since  $t_{\gamma_0}(\alpha_0) \leq_{\mathbb{T}} t_0 <_{\mathbb{T}} t_1, t_{\gamma_1}(\alpha_0) \leq_{\mathbb{T}} t_1$  and

$$\text{lh}_{\mathbb{T}}(t_{\gamma_0}(\alpha_0)) = \gamma_0 < \gamma_1 = \text{lh}_{\mathbb{T}}(t_{\gamma_1}(\alpha_0)),$$

we can conclude that  $t_{\gamma_0}(\alpha_0) <_{\mathbb{T}} t_{\gamma_1}(\alpha_0)$  holds.

Given  $\gamma < \theta$  and  $\alpha < \lambda$ , let  $\delta \in U \setminus \gamma$  be minimal with  $A_\delta \subseteq \mathbb{T}_{<\delta} \times \mu$  for all  $\bar{\delta} \in U \cap \delta$  and define  $b_\gamma(\alpha)$  to be the unique element of  $\mathbb{T}(\gamma)$  with  $b_\gamma(\alpha) \leq_{\mathbb{T}} t_\gamma(\alpha)$ . By the above computations, if  $\bar{\gamma} < \gamma < \theta$ , then there is  $\alpha < \lambda$  with  $b_{\bar{\gamma}}(\alpha) <_{\mathbb{T}} b_\gamma(\alpha)$ . This shows that the resulting sequence  $\langle b_\gamma : \lambda \longrightarrow \mathbb{T}(\gamma) \mid \gamma < \theta \rangle$  is an ascending path of width less than  $\kappa$  through  $\mathbb{T}$ .  $\square$

*Proof of Theorem 1.9.* Let  $\kappa < \theta$  be a regular cardinal with  $\mu^{<\kappa} < \theta$  for all  $\mu < \theta$ . If forcing with the partial order  $\mathbb{P}_\kappa(\mathbb{T})$  does not preserve the regularity of  $\theta$ , then  $\mathbb{P}_\kappa(\mathbb{T})$  does not satisfy the  $\theta$ -chain condition and Theorem 1.11 shows that there is an ascending path of width less than  $\kappa$  through  $\mathbb{T}$ . In the other direction, assume that there is an ascending path of width less than  $\kappa$  through  $\mathbb{T}$  and forcing with  $\mathbb{P}_\kappa(\mathbb{T})$  preserves the regularity of  $\theta$ . Let  $G$  be  $\mathbb{P}_\kappa(\mathbb{T})$ -generic over  $V$ . In  $V[G]$ ,  $\mathbb{T}$  contains an ascending path of width less than  $\kappa$  and  $\theta$  is not the successor of a cardinal of cofinality less than  $\kappa$ . In this situation, Corollary 1.7 implies that  $\mathbb{T}$  is not special in  $V[G]$ , a contradiction.  $\square$

Note that the proof of Theorem 1.11 shows that, if  $\kappa < \theta$  is a regular cardinal with  $\mu^{<\kappa} < \theta$  for all  $\mu < \theta$  and there is an ascending path of width  $\lambda < \kappa$  through  $\mathbb{T}$ , then there is such a path  $\langle b_\gamma : \lambda \longrightarrow \mathbb{T}(\gamma) \mid \gamma < \theta \rangle$  with the additional property that for all  $\bar{\gamma} < \gamma < \theta$  there is an  $\alpha < \lambda$  with  $b_{\bar{\gamma}}(\alpha) <_{\mathbb{T}} b_\gamma(\alpha)$ . In the general terminology of [4], such a path is called an  $\mathcal{F}_\lambda$ -ascent path through  $\mathbb{T}$ . It is not known to the author whether the existence of a  $\mathcal{F}_\lambda$ -ascent path is always equivalent to the existence of an ascending path of width  $\lambda$  (see Question 6.5).

## 3. NARROW SYSTEM AND ULTRAFILTERS

In this section, we will prove the non-existence results stated in Lemma 1.4 and Theorem 1.13 in Section 1. Instead of presenting the author's original proofs of these statements, we will present more elegant arguments that rely on the notion of *narrow systems* introduced by Magidor and Shelah in [15] and recent results of Lambie-Hanson contained in [12].

**Definition 3.1** ([15, Definition 2.2]). Let  $\vartheta$  be a limit ordinal, let  $D$  be an unbounded subset of  $\vartheta$  and let  $\lambda > 0$  be a cardinal.

- (i) A set  $\mathcal{R}$  of binary transitive relations on  $D \times \lambda$  is a  $\vartheta$ -system of width  $\lambda$  on  $D$  if the following statements hold:
  - (a) If  $\gamma, \bar{\gamma} \in D$ ,  $\alpha, \bar{\alpha} < \lambda$  and  $R \in \mathcal{R}$  with  $\langle \bar{\gamma}, \bar{\alpha} \rangle R \langle \gamma, \alpha \rangle$ , then  $\bar{\gamma} < \gamma$ .
  - (b) If  $\gamma, \gamma_0, \gamma_1 \in D$ ,  $\alpha, \alpha_0, \alpha_1 < \lambda$  and  $R \in \mathcal{R}$  such that  $\gamma_0 < \gamma_1$  and  $\langle \gamma_i, \alpha_i \rangle R \langle \gamma, \alpha \rangle$  for all  $i < 2$ , then  $\langle \gamma_0, \alpha_0 \rangle R \langle \gamma_1, \alpha_1 \rangle$ .
  - (c) If  $\gamma, \bar{\gamma} \in D$  with  $\bar{\gamma} < \gamma$ , then there are  $\alpha, \bar{\alpha} < \lambda$  and  $R \in \mathcal{R}$  with  $\langle \bar{\gamma}, \bar{\alpha} \rangle R \langle \gamma, \alpha \rangle$ .
- (ii) A  $\vartheta$ -system  $\mathcal{R}$  of width  $\lambda$  is *narrow* if  $|\mathcal{R}| < \lambda^+ < |\vartheta|$ .
- (iii) Given a  $\vartheta$ -system  $\mathcal{R}$  of width  $\lambda$  on  $D$  and  $R \in \mathcal{R}$ , a subset  $B$  of  $D \times \lambda$  is an *R-branch through  $\mathcal{R}$*  if for all  $\langle \gamma_0, \alpha_0 \rangle, \langle \gamma_1, \alpha_1 \rangle \in B$  with  $\langle \gamma_0, \alpha_0 \rangle \neq \langle \gamma_1, \alpha_1 \rangle$  there is an  $i < 2$  with  $\langle \gamma_i, \alpha_i \rangle R \langle \gamma_{1-i}, \alpha_{1-i} \rangle$ .
- (iv) Given a  $\vartheta$ -system  $\mathcal{R}$  of width  $\lambda$  on  $D$  and  $R \in \mathcal{R}$ , an *R-branch  $B$  through  $\mathcal{R}$  is cofinal* if the set  $\{\gamma \in D \mid \exists \alpha < \lambda \langle \gamma, \alpha \rangle \in B\}$  is unbounded in  $\vartheta$ .
- (v) Given a  $\vartheta$ -system  $\mathcal{R}$  of width  $\lambda$  on  $D$  and a set  $\mathcal{B}$  of cardinality at most  $\lambda$  with the property that every element of  $\mathcal{B}$  is an *R-branch through  $\mathcal{R}$*  for some  $R \in \mathcal{R}$ , then  $\mathcal{B}$  is a *full set of branches through  $\mathcal{R}$*  if for every  $\gamma \in D$  there are  $B \in \mathcal{B}$  and  $\alpha < \lambda$  with  $\langle \gamma, \alpha \rangle \in B$ .

A simple cardinality argument shows that the existence of a full set of branches through a narrow  $\theta$ -system  $\mathcal{R}$  implies the existence of a cofinal branch through  $\mathcal{R}$ .

It is easy to see that, if the tree  $\mathbb{T}$  has no cofinal branches and there is an ascending path of width  $\lambda$  through  $\mathbb{T}$ , then there is a  $\theta$ -system  $\mathcal{R}$  of width  $\lambda$  such that  $|\mathcal{R}| = 1$  and there are no cofinal branches through  $\mathcal{R}$ . This shows that the statements of Lemma 1.4 and Theorem 1.13 follow from the next lemma. Note that many of the statements of the lemma already appear in [12]. For sake of completeness, we also present the proofs of these results in this paper.

**Lemma 3.2.** (i) *A narrow  $\theta$ -system of finite width has a full set of branches.*

- (ii) *If  $\theta$  is weakly compact, then every narrow  $\theta$ -system has a full set of branches.*
- (iii) *If  $\kappa \leq \theta$  is  $\theta$ -compact, then every narrow  $\theta$ -system of width less than  $\kappa$  has a full set of branches.*
- (iv) *If  $\kappa \leq \theta$  is indestructibly weakly compact, then every narrow  $\theta$ -system of width less than  $\kappa$  has a cofinal branch.*
- (v) *If  $\theta$  is weakly compact,  $\kappa < \theta$  is an uncountable regular cardinal and  $G$  is  $\text{Col}(\kappa, < \theta)$ -generic over  $V$ , then in  $V[G]$  every narrow  $\theta$ -system has a cofinal branch.*
- (vi) *If  $\kappa \leq \theta$  is  $\theta$ -compact,  $\nu < \kappa$  is an uncountable regular cardinal and  $G$  is  $\text{Col}(\nu, < \kappa)$ -generic over  $V$ , then in  $V[G]$  every narrow  $\theta$ -system of width less than  $\nu$  has a cofinal branch.*

The proof of the last three statements relies on the following preservation lemma proven by Lambie-Hanson in [12].

**Lemma 3.3** ([12, Lemma 4.1]). *Let  $\theta$  be an uncountable regular cardinal, let  $\kappa < \theta$  be a regular cardinal, let  $D$  be an unbounded subset of  $\theta$  and let  $\mathcal{R}$  be a narrow  $\theta$ -system of width less than  $\kappa$  on  $D$ . If there is a  $<\kappa$ -closed partial order  $\mathbb{P}$  with the property that in some  $\mathbb{P}$ -generic extension there is a complete set of branches through  $\mathcal{R}$ , then there is a cofinal branch through  $\mathcal{R}$  in  $V$ .*

The idea used in the proof of the next lemma is taken from Baumgartner's elegant proof of [3, Theorem 8.2]. Remember that, given a collection  $\mathcal{S}$  of subsets of some set  $D$ , a filter  $\mathcal{F}$  on  $D$  is an  $\mathcal{S}$ -ultrafilter if for all  $S \in \mathcal{S}$ , either  $S \in \mathcal{F}$  or  $D \setminus S \in \mathcal{F}$  holds.

**Lemma 3.4.** *Let  $\theta$  be an uncountable regular cardinal and let  $\kappa < \theta$  be a cardinal. Then for every narrow  $\theta$ -system  $\mathcal{R}$  of width less than  $\kappa$  there is a collection  $\mathcal{S}_{\mathcal{R}}$  of  $\theta$ -many subsets of  $\theta$  such that the following statements hold:*

- (i) *If there is a  $<\kappa$ -closed  $\mathcal{S}_{\mathcal{R}}$ -ultrafilter on  $\theta$  that consists of unbounded subsets of  $\theta$ , then there is a full set of branches through  $\mathcal{R}$ .*
- (ii) *If there is a  $<\kappa$ -closed partial order  $\mathbb{P}$  with the property that forcing with  $\mathbb{P}$  adds a  $<\kappa$ -closed  $\mathcal{S}_{\mathcal{R}}$ -ultrafilter on  $\theta$  that consists of unbounded subsets of  $\theta$ , then  $\mathcal{R}$  has a cofinal branch in  $V$ .*

*Proof.* Let  $D$  be an unbounded subset of  $\theta$ , let  $0 < \lambda < \kappa$  be a cardinal and let  $\mathcal{R}$  be a narrow  $\theta$ -system of width  $\lambda$  on  $D$ . Set  $\mathcal{I} = \lambda \times \lambda \times \mathcal{R}$ . Define  $\mathcal{S}_{\mathcal{R}}$  to be the set consisting of all subsets of  $D$  of the form

$$S_{\gamma, I} = \{\delta \in D \mid \langle \gamma, \alpha_I \rangle R_I \langle \delta, \beta_I \rangle\}$$

for some  $\gamma \in D$  and  $I = \langle \alpha_I, \beta_I, R_I \rangle \in \mathcal{I}$ . Then  $\mathcal{S}_{\mathcal{R}}$  has cardinality  $\theta$ .

Both of the above assumptions imply that there is a  $<\kappa$ -closed partial order  $\mathbb{P}$ , a filter  $\mathcal{G}$  on  $\mathbb{P}$  that is generic over  $V$  and a  $<\kappa$ -closed  $\mathcal{S}_{\mathcal{R}}$ -ultrafilter  $\mathcal{F}$  on  $D$  in  $V[G]$  that consists of unbounded subsets of  $\theta$ . Work in  $V[G]$ . Given  $\gamma \in D$ , we have

$$D \setminus (\gamma + 1) = \bigcup \{S_{\gamma, I} \mid I \in \mathcal{I}\}.$$

In this situation, our assumptions on  $\mathcal{F}$  imply that there is a sequence  $\langle I_\gamma \mid \gamma \in D \rangle$  with the property that  $S_{\gamma, I_\gamma} \in \mathcal{F}$  for all  $\gamma \in D$ . Given  $I \in \mathcal{I}$ , define

$$B_I = \{\langle \gamma, \alpha_I \rangle \mid \gamma \in D, I = I_\gamma\}.$$

Pick  $I \in \mathcal{I}$  and  $\gamma_0, \gamma_1 \in D$  with  $\gamma_0 < \gamma_1$  and  $I = I_{\gamma_0} = I_{\gamma_1}$ . Then we have  $S = S_{\gamma_0, I} \cap S_{\gamma_1, I} \in \mathcal{F}$  and our assumptions imply that there is a  $\delta \in S$  with  $\delta > \gamma_i$  for all  $i < 2$ . This implies that  $\langle \gamma_i, \alpha_I \rangle R_I \langle \delta, \beta_I \rangle$  for all  $i < 2$ . By the definition of narrow systems, this implies that  $\langle \gamma_0, \alpha_I \rangle R_I \langle \gamma_1, \alpha_I \rangle$ .

The above computations show that  $B_I$  is an  $R_I$ -branch through  $\mathcal{R}$  for every  $I \in \mathcal{I}$ . Since we have  $\langle \gamma, \alpha_{I_\gamma} \rangle \in B_{I_\gamma}$  for all  $\gamma \in D$ , the set  $\{B_I \mid I \in \mathcal{I}\}$  is a full set of branches through  $\mathcal{R}$  in  $V[G]$ . By Lemma 3.3, this implies that there is a cofinal branch through  $\mathcal{R}$  in  $V$ .  $\square$

*Proof of Lemma 3.2.* (i) Let  $\mathcal{U}$  be an ultrafilter on  $\theta$  that extends the filter of cobounded subsets of  $\theta$ . Then  $\mathcal{U}$  is closed under finite intersections and consists of unbounded subsets of  $\theta$ . If  $\mathcal{R}$  is a narrow  $\theta$ -system of finite width, then  $\mathcal{U}$  is an  $\mathcal{S}_{\mathcal{R}}$ -ultrafilter and we can use the first part of Lemma 3.4 to conclude that there is a full set of branches through  $\mathcal{R}$ .

(ii) Assume that  $\theta$  is weakly compact and let  $\mathcal{R}$  be a narrow  $\theta$ -system. By the filter property of weakly compact cardinals, there is a non-principal  $<\kappa$ -closed  $\mathcal{S}_{\mathcal{R}}$ -ultrafilter on  $\theta$  and the above statement follows directly from the first part of Lemma 3.4.

(iii) Assume that  $\kappa \leq \theta$  is  $\theta$ -compact. By the filter property of  $\theta$ -compact cardinals, there is a  $<\kappa$ -closed ultrafilter on  $\theta$  that extends the filter of all cobounded subsets of  $\theta$ . In this situation, the first part of Lemma 3.4 implies that every narrow  $\theta$ -system of width less than  $\kappa$  has a full set of branches.

(iv) Assume that  $\kappa \leq \theta$  is indestructibly weakly compact, let  $\mathcal{R}$  be a narrow  $\theta$ -system of width less than  $\kappa$  and let  $G$  be  $\text{Col}(\kappa, \theta)$ -generic over  $V$ . Then  $\kappa$  is weakly compact in  $V[G]$  and the proof of the second part of the lemma shows that there is a non-principal  $<\kappa$ -closed  $\mathcal{S}_{\mathcal{R}}$ -ultrafilter in  $V[G]$ . In this situation, the second part of Lemma 3.4 shows that there is a cofinal branch through  $\mathcal{R}$  in  $V$ .

(v) Assume that  $\theta$  is weakly compact,  $\kappa < \theta$  is an uncountable regular cardinal and  $G$  is  $\text{Col}(\kappa, <\theta)$ -generic over  $V$ . Let  $\mathcal{R}$  be a narrow  $\theta$ -system in  $V[G]$  and let  $\mathcal{S}_{\mathcal{R}}$  denote the corresponding collection of subsets of  $\theta$  given by Lemma 3.4. Since the partial order  $\text{Col}(\kappa, <\theta)$  satisfies the  $\theta$ -chain condition, there is a  $\text{Col}(\kappa, <\kappa)$ -name  $\dot{S} \in H(\theta^+)^V$  with  $\mathcal{S}_{\mathcal{R}} = \dot{S}$ .

Work in  $V$  and pick an elementary submodel  $M$  of  $H(\theta^+)$  of cardinality  $\theta$  with  $\dot{S} \in M$  and  ${}^{<\theta}M \subseteq M$ . By the embedding property of weakly compact cardinals, there is a transitive set  $N$  of cardinality  $\theta$  with  ${}^{<\theta}N \subseteq N$  and an elementary embedding  $j : M \rightarrow N$  with critical point  $\theta$ . Then  $\text{Col}(\kappa, <\theta)$  is an element of  $M$  and  $j(\text{Col}(\kappa, <\theta))$  is isomorphic to  $\text{Col}(\kappa, <\theta) \times \text{Col}(\kappa, [\theta, j(\theta)])$  both in  $V$  and  $N$ .

Let  $H$  be  $\text{Col}(\kappa, [\theta, j(\theta)])$ -generic over  $V[G]$ . Then we can lift  $j$  to an elementary embedding  $j_* : M[G] \rightarrow N[G, H]$ . Let  $\mathcal{F} = \{A \in \mathcal{P}(\theta) \cap M[G] \mid \theta \in j_*(A)\}$  be the induced  $M[G, H]$ -ultrafilter in  $V[G, H]$ . Since  $\text{Col}(\kappa, [\theta, j(\theta)])$  is  $<\kappa$ -closed and  $\mathcal{S}_{\mathcal{R}} \in M[G]$ , the filter  $\mathcal{F}$  is a non-principal  $<\kappa$ -closed  $\mathcal{S}_{\mathcal{R}}$ -ultrafilter on  $\theta$  in  $V[G, H]$  and the second part of Lemma 3.4 shows that  $\mathcal{R}$  has a cofinal branch in  $V[G]$ .

(vi) Assume that  $\kappa \leq \theta$  is  $\theta$ -compact,  $\nu < \kappa$  is an uncountable regular cardinal and  $G$  is  $\text{Col}(\nu, <\kappa)$ -generic over  $V$ . In  $V$ , the  $\theta$ -compactness of  $\kappa$  yields an elementary embedding  $j : V \rightarrow M$  with critical point  $\kappa$  such that  $j[\theta] \subseteq \lambda$  for some  $\lambda < j(\theta)$ . Then  $j(\text{Col}(\kappa, <\theta))$  is isomorphic to  $\text{Col}(\kappa, <\theta) \times \text{Col}(\kappa, [\theta, j(\theta)])$  both in  $V$  and  $M$ . Let  $H$  be  $\text{Col}(\kappa, [\theta, j(\theta)])$ -generic over  $V[G]$ . Then we can lift  $j$  to an elementary embedding  $j_* : V[G] \rightarrow M[G, H]$  in  $V[G, H]$ . Define  $\mathcal{F} = \{A \in \mathcal{P}(\theta)^{V[G]} \mid \lambda \in j_*(A)\}$ . Since  $\text{Col}(\kappa, [\theta, j(\theta)])$  is  $<\kappa$ -closed, we can conclude that  $\mathcal{F}$  is a  $<\kappa$ -closed  $V[G]$ -ultrafilter on  $\theta$  in  $V[G, H]$  that consists of unbounded subsets of  $\theta$ . With the help of the second part of Lemma 3.4, we can conclude that in  $V[G]$  every narrow  $\theta$ -system has a cofinal branch.  $\square$

As mentioned in Section 1, the results of [12] also provide a global non-existence result for trees without cofinal branches containing ascending paths of small width. In the proof of [12, Theorem 5.2], Lambie-Hanson starts with a model containing a proper class of supercompact cardinals and produces a class forcing extension in which every narrow  $\vartheta$ -system for an uncountable regular cardinal  $\vartheta$  has a cofinal branch. By the above remarks, every tree of uncountable regular height  $\vartheta$  containing an ascending path of width  $\lambda$  with  $\lambda^+ < \vartheta$  has a cofinal branch in this model.

## 4. TREES CONSTRUCTED FROM WALKS ON ORDINALS

This section contains the proofs of Theorem 1.10 and Theorem 1.12. These proofs show that certain combinatorial principles allow us to construct trees with very specific properties. The construction of these trees uses the concept of *walks on ordinals* and their characteristics introduced by Todorčević.

**Definition 4.1** (Todorčević). (i) A sequence  $\vec{C} = \langle C_\gamma \subseteq \gamma \mid \gamma < \theta \rangle$  is a *C-sequence of length  $\theta$*  if the following statements hold for all  $\gamma < \theta$ .

- (a) If  $\gamma$  is a limit ordinal, then  $C_\gamma$  is a closed unbounded subset of  $\gamma$ .
- (b) If  $\gamma = \bar{\gamma} + 1$ , then  $C_\gamma = \{\bar{\gamma}\}$ .

(ii) Let  $\vec{C} = \langle C_\gamma \mid \gamma < \theta \rangle$  be a C-sequence of length  $\theta$ .

- (a) Given  $\gamma \leq \delta < \theta$ , the *walk from  $\delta$  to  $\gamma$  through  $\vec{C}$*  is the unique sequence  $\langle \varepsilon_0, \dots, \varepsilon_n \rangle$  with  $\varepsilon_0 = \delta$ ,  $\varepsilon_n = \gamma$  and  $\varepsilon_{i+1} = \min(C_{\varepsilon_i} \setminus \gamma)$  for all  $i < n$ . In this situation, we define the *full code of the walk from  $\delta$  to  $\gamma$  through  $\vec{C}$*  to be the sequence

$$\rho_0^{\vec{C}}(\gamma, \delta) = \langle \text{otp}(C_{\varepsilon_0} \cap \gamma), \dots, \text{otp}(C_{\varepsilon_{n-1}} \cap \gamma) \rangle.$$

- (b) Given  $\delta < \kappa$ , we define

$$\rho_0^{\vec{C}}(\cdot, \delta) : \delta + 1 \longrightarrow {}^{<\omega}\theta; \gamma \longmapsto \rho_0^{\vec{C}}(\gamma, \delta).$$

- (c) We define  $\mathbb{T}(\rho_0^{\vec{C}})$  to be the tree of height  $\theta$  consisting of all functions of the form  $\rho_0^{\vec{C}}(\cdot, \delta) \upharpoonright \gamma$  with  $\gamma \leq \delta < \theta$  ordered by inclusion.

Remember that a tree  $\mathbb{T}$  is a  $\theta$ -Aronszajn tree if  $\mathbb{T}$  has no cofinal branches and  $|\mathbb{T}(\gamma)| < \theta$  holds for all  $\gamma < \theta$ . The following results of Todorčević show how such sequences can be used to construct  $\theta$ -Aronszajn trees.

**Lemma 4.2** ([24, Lemma 1.3]). *If  $\vec{C} = \langle C_\gamma \mid \gamma < \theta \rangle$  is a C-sequence and  $\gamma < \theta$ , then  $|\mathbb{T}(\rho_0^{\vec{C}})(\gamma)| \leq |\{C_\delta \cap \gamma \mid \gamma \leq \delta < \theta\}| + \aleph_0$ .*

**Lemma 4.3** ([24, Lemma 1.7]). *The following statements are equivalent for every C-sequence  $\vec{C} = \langle C_\gamma \mid \gamma < \theta \rangle$  of length  $\theta$ .*

- (i) *There is a cofinal branches through the tree  $\mathbb{T}(\rho_0^{\vec{C}})$ .*
- (ii) *There is a club subset  $C$  of  $\theta$  and  $\xi < \theta$  such that for all  $\xi \leq \gamma < \theta$  there is  $\gamma \leq \delta < \theta$  with  $C \cap \gamma = C_\delta \cap [\xi, \gamma)$ .*

We will now show that trees of the form  $\mathbb{T}(\rho_0^{\vec{C}})$  do not contain ascending paths of small width if they contain no cofinal branches and the underlying sequence  $\vec{C}$  is coherent in the following sense.

**Definition 4.4.** Given  $S \subseteq \theta$ , we say that a C-sequence  $\vec{C} = \langle C_\gamma \mid \gamma < \theta \rangle$  is *S-coherent* if  $C_{\bar{\gamma}} = C_\gamma \cap \bar{\gamma}$  holds for all  $\gamma \in \theta \cap \text{Lim}$  and  $\bar{\gamma} \in \text{Lim}(C_\gamma) \cap S$ .

The proof of the following lemma relies on some computations of full codes of walks that are presented in detail in [9, Section 3].

**Lemma 4.5.** *Let  $\lambda < \theta$  be a cardinal, let  $S \subseteq S_{>\lambda}^\theta$  be stationary in  $\theta$  and let  $\vec{C}$  be an S-coherent C-sequence of length  $\theta$ . If the tree  $\mathbb{T}(\rho_0^{\vec{C}})$  contains an ascending path of width  $\lambda$ , then  $\mathbb{T}(\rho_0^{\vec{C}})$  has a cofinal branch.*

*Proof.* Let  $\vec{C} = \langle C_\gamma \mid \gamma < \theta \rangle$ . Set  $\mathbb{T} = \mathbb{T}(\rho_0^{\vec{C}})$  and  $\rho_0 = \rho_0^{\vec{C}}$ . Assume, towards a contradiction, that  $\langle b_\gamma : \lambda \longrightarrow \mathbb{T}(\gamma) \mid \gamma < \theta \rangle$  is an ascending path through  $\mathbb{T}$ .

Given  $\gamma \in S$  and  $\alpha < \lambda$ , let  $\gamma \leq \delta_\gamma^\alpha < \theta$  be minimal with  $b_\gamma(\alpha) = \rho_0(\cdot, \delta_\gamma^\alpha) \upharpoonright \gamma$ , let  $\langle \varepsilon_\gamma^\alpha(0), \dots, \varepsilon_\gamma^\alpha(n_\gamma^\alpha) \rangle$  denote the walk from  $\delta_\gamma^\alpha$  to  $\gamma$  through  $\vec{C}$  and let  $k_\gamma^\alpha \leq n_\gamma^\alpha$  be minimal with  $C_{\varepsilon_\gamma^\alpha(k_\gamma^\alpha)} \cap \gamma$  unbounded in  $\gamma$ . By our assumptions, we can find  $\mu \leq \nu < \theta$  and  $E \subseteq S \setminus (\nu + 1)$  stationary in  $\theta$  such that  $\max(C_{\varepsilon_\gamma^\alpha(l)} \cap \gamma) < \mu$  and  $\min(C_\gamma \setminus \mu) = \nu$  for all  $\gamma \in E$ ,  $\alpha < \lambda$  and  $l < k_\gamma^\alpha$ .

**Claim.** *If  $\gamma_0, \gamma_1 \in E$  and  $\alpha < \lambda$  with  $\gamma_0 < \gamma_1$  and  $b_{\gamma_0}(\alpha) <_{\mathbb{T}} b_{\gamma_1}(\alpha)$ , then  $k_{\gamma_0}^\alpha = k_{\gamma_1}^\alpha$ .*

*Proof of the Claim.* Note that our assumptions imply that  $\nu \in C_{\varepsilon_{\gamma_i}^\alpha(k_{\gamma_i}^\alpha)}$  for all  $i < 2$ . Given  $i < 2$ , we can combine this observation with [9, Lemma 3.2] to see that the sequence  $\langle \varepsilon_{\gamma_i}^\alpha(0), \dots, \varepsilon_{\gamma_i}^\alpha(k_{\gamma_i}^\alpha), \nu \rangle$  is the walk from  $\delta_{\gamma_i}^\alpha$  to  $\nu$  through  $\vec{C}$ . Since our assumptions imply that  $\rho_0(\nu, \delta_{\gamma_0}^\alpha) = \rho_0(\nu, \delta_{\gamma_1}^\alpha)$  holds, we can conclude that

$$k_{\gamma_0}^\alpha = \text{lh}(\rho_0(\nu, \delta_{\gamma_0}^\alpha)) - 1 = \text{lh}(\rho_0(\nu, \delta_{\gamma_1}^\alpha)) - 1 = k_{\gamma_1}^\alpha. \quad \square$$

**Claim.** *If  $\gamma_0, \gamma_1 \in E$  with  $\gamma_0 < \gamma_1$ , then  $C_{\gamma_0} = C_{\gamma_1} \cap \gamma_0$ .*

*Proof of the Claim.* By our assumption, there is an  $\alpha < \lambda$  with  $b_{\gamma_0}(\alpha) <_{\mathbb{T}} b_{\gamma_1}(\alpha)$ . In this situation, the above claim shows that  $k_{\gamma_0}^\alpha = k_{\gamma_1}^\alpha$ . Set  $k = k_{\gamma_0}^\alpha$  and pick an ordinal  $\xi \in C_{\gamma_0} \cap [\mu, \gamma_0)$ . Since our assumptions imply that  $C_{\gamma_0} = C_{\varepsilon_{\gamma_0}^\alpha(k)} \cap \gamma_0$  and hence  $\xi \in C_{\varepsilon_{\gamma_0}^\alpha(k)}$ , an application of [9, Lemma 3.2] shows that the sequence  $\langle \varepsilon_{\gamma_0}^\alpha(0), \dots, \varepsilon_{\gamma_0}^\alpha(k), \xi \rangle$  is the walk from  $\delta_{\gamma_0}^\alpha$  to  $\xi$  through  $\vec{C}$ . Another application of [9, Lemma 3.2] shows that the sequence  $\langle \varepsilon_{\gamma_1}^\alpha(0), \dots, \varepsilon_{\gamma_1}^\alpha(k) \rangle$  is an initial segment of the walk from  $\delta_{\gamma_1}^\alpha$  to  $\xi$ . In this situation,  $\rho_0(\xi, \delta_{\gamma_0}^\alpha) = \rho_0(\xi, \delta_{\gamma_1}^\alpha)$  implies that the sequence  $\langle \varepsilon_{\gamma_1}^\alpha(0), \dots, \varepsilon_{\gamma_1}^\alpha(k), \xi \rangle$  is the walk from  $\delta_{\gamma_1}^\alpha$  to  $\xi$  through  $\vec{C}$ . In particular, this shows that  $\xi$  is an element of  $C_{\varepsilon_{\gamma_1}^\alpha}$ . Since our assumptions also imply that  $C_{\gamma_1} = C_{\varepsilon_{\gamma_1}^\alpha} \cap \gamma_1$ , we can conclude that  $\xi \in C_{\gamma_1}$ . This shows that  $C_{\gamma_0} \subseteq C_{\gamma_1}$  and hence we have  $\gamma_0 \in \text{Lim}(C_{\gamma_1}) \cap S$  and  $S$ -coherency implies  $C_{\gamma_0} = C_{\gamma_1} \cap \gamma_0$ .  $\square$

The last claim shows that  $C = \bigcup \{C_\gamma \mid \gamma \in E\}$  is a club in  $\theta$  with  $C_\gamma = C \cap \gamma$  for all  $\gamma \in E$ . Using Lemma 4.3, we can conclude that  $\mathbb{T}$  has a cofinal branch.  $\square$

We will later show that it is not possible to prove the statement of the above lemma without some coherency assumption on the  $C$ -sequence (see Theorem 4.12).

**Definition 4.6** (Todorćević). A  $C$ -sequence  $\vec{C} = \langle C_\gamma \mid \gamma < \theta \rangle$  is a  $\square(\theta)$ -sequence if  $\vec{C}$  is  $\theta$ -coherent and there is no club  $C$  in  $\theta$  with  $C_\gamma = C \cap \gamma$  for all  $\gamma \in \text{Lim}(C)$ .

If  $\vec{C}$  is a  $\square(\theta)$ -sequence, then Lemma 4.3 and Lemma 4.2 imply that  $\mathbb{T}(\rho_0^{\vec{C}})$  is a  $\theta$ -Aronszajn tree. In combination with Lemma 4.5, this yields the following result.

**Corollary 4.7.** *If  $\vec{C}$  is  $\square(\theta)$ -sequence and  $\lambda$  is a cardinal with  $\lambda^+ < \theta$ , then the tree  $\mathbb{T}(\rho_0^{\vec{C}})$  does not contain an ascending path of width  $\lambda$ .*  $\square$

Seminal results of Jensen and Todorćević (see [8, Section 6] and [24, Theorem 1.10]) show that the existence of a constructible  $\square(\theta)$ -sequence follows from the assumption that  $\theta$  is not weakly compact in  $L$ . In combination with the main result of [25], these results allow us to prove Theorem 1.10.

*Proof of Theorem 1.10.* Let  $\kappa$  be an uncountable cardinal such that  $\kappa = \kappa^{<\kappa}$  and  $\kappa^+$  is not weakly compact in  $L$ . In this situation, [24, Theorem 1.10] implies that there is a  $\square(\kappa^+)$ -sequence and [25, Theorem 3] implies that there is a  $\square(\kappa^+)$ -sequence  $\vec{C}$  with the property that the tree  $\mathbb{T} = \mathbb{T}(\rho_0^{\vec{C}})$  is not special. Then the partial order  $\mathbb{P}_\kappa(\mathbb{T})$  is  $<\kappa$ -closed and well-met. Moreover, Corollary 4.7 shows that there are no ascending paths of width less than  $\kappa$  through  $\mathbb{T}$  and, by our assumptions on  $\kappa$ , we can apply Theorem 1.9 to conclude that  $\mathbb{P}_\kappa(\mathbb{T})$  satisfies the  $\kappa^+$ -chain condition. Given  $t \in \mathbb{T}$ , define  $D_t$  to be the dense subset  $\{p \in \mathbb{P}_\kappa(\mathbb{T}) \mid t \in \text{dom}(p)\}$  of  $\mathbb{P}_\kappa(\mathbb{T})$ . Since  $\mathbb{T}$  is a  $\kappa^+$ -Aronszajn tree, the collection  $\mathcal{D} = \{D_t \mid t \in \mathbb{T}\}$  has cardinality  $\kappa^+$ . But then there is no  $\mathcal{D}$ -generic filter on  $\mathbb{P}_\kappa(\mathbb{T})$ , because the existence of such a filter would imply that  $\mathbb{T}$  is special. This shows that  $\text{FA}_{\kappa^+}(\mathbb{P}_\kappa(\mathbb{T}))$  fails.  $\square$

In the remainder of this section, we will prove Theorem 1.12. The following definition and lemmas are crucial for the proof of this result.

**Definition 4.8.** A  $C$ -sequence  $\vec{C} = \langle C_\gamma \mid \gamma < \theta \rangle$  avoids  $S \subseteq \theta$  if  $\text{Lim}(C_\gamma) \cap S = \emptyset$  holds for all  $\gamma \in \text{Lim} \cap \theta$ .

**Lemma 4.9** ([5, Lemma 6.8]). *There is a club  $D$  in  $\theta$  with the property that whenever  $\vec{C}$  is a  $C$ -sequence of length  $\theta$  that avoids a subset  $S$  of  $\theta$ , then  $D \cap S$  is non-stationary with respect to  $\mathbb{T}(\rho_0^{\vec{C}})$ .*

**Lemma 4.10** ([5, Lemma 6.4]). *Assume that  $\mathbb{T}$  is a  $\theta$ -Aronszajn tree that does not split at limit levels. If there is a stationary subset  $S$  of  $\theta$  such that  $S$  is non-stationary with respect to  $\mathbb{T}$ , then the partial order  $\mathbb{P}_\omega(\mathbb{T})$  is  $\theta$ -Knaster.*

Note that trees of the form  $\mathbb{T}(\rho_0^{\vec{C}})$  for some  $C$ -sequence  $\vec{C}$  consist of functions ordered by inclusion and therefore such trees do not split at limit levels. In the following, we will start from the assumption that there is a  $\square(\theta)$ -sequence that avoids a stationary set consisting of ordinals of small cofinality and use the above lemma to find a  $C$ -sequence  $\vec{C}$  of length  $\theta$  with the property that the tree  $\mathbb{T}(\rho_0^{\vec{C}})$  is a  $\theta$ -Aronszajn tree containing an ascending path of small width and the property that the partial order  $\mathbb{P}_\omega(\mathbb{T}(\rho_0^{\vec{C}}))$  is  $\theta$ -Knaster. The constructed path will satisfy the following stronger property first considered by Laver (see [7] and [22]). In the general terminology of [4], such paths are called  $\mathcal{F}_\lambda^{bd}$ -ascent paths.

**Definition 4.11** (Laver). Let  $\lambda > 0$  be a cardinal. A  $\lambda$ -ascent paths through  $\mathbb{T}$  is a sequence  $\langle b_\gamma : \lambda \rightarrow \mathbb{T}(\gamma) \mid \gamma < \theta \rangle$  with the property that for all  $\gamma, \tilde{\gamma} < \theta$ , there is a  $\tilde{\lambda} < \lambda$  such that  $b_{\tilde{\lambda}}(\alpha) \leq_{\mathbb{T}} b_\gamma(\alpha)$  for all  $\tilde{\lambda} < \alpha < \lambda$ .

In [25], Todorćević constructed a  $\theta$ -Aronszajn tree containing such a  $\kappa$ -ascent path assuming the existence of a  $\square(\theta)$ -sequence  $\vec{C} = \langle C_\gamma \mid \gamma < \theta \rangle$  with the property that there is an infinite regular cardinal  $\kappa < \theta$  such that  $\theta$  is not the successor of a cardinal of cofinality  $\kappa$  and the set  $\{\gamma < \theta \mid \text{otp}(C_\gamma) = \kappa\}$  is stationary in  $\theta$ . The results of [11, Section 3] show that this assumption is slightly stronger than the assumptions of the next theorem. This theorem also shows that the conclusion of Lemma 4.5 does not hold without some coherency assumptions on the sequence  $\vec{C}$ .

**Theorem 4.12.** *Let  $\lambda < \theta$  be an infinite regular cardinal and let  $S \subseteq S_\lambda^\theta$  be stationary in  $\theta$ . Assume that there is a  $\square(\theta)$ -sequence that avoids  $S$ . Then there is a  $C$ -sequence  $\vec{C}$  of length  $\theta$  that avoids  $S$  and has the property that the tree  $\mathbb{T}(\rho_0^{\vec{C}})$  is a  $\theta$ -Aronszajn tree that contains a  $\lambda$ -ascent path.*

*Proof.* Fix a  $\square(\theta)$ -sequence  $\vec{D} = \langle D_\gamma \mid \gamma < \theta \rangle$  that avoids  $S$  and let  $A$  denote the set of all ordinals less than  $\theta$  that are divisible by  $\omega \cdot \lambda$ , i.e.  $A = \{\omega \cdot \lambda \cdot \gamma \mid \gamma < \theta\}$ . Then  $0 \in A$  and  $A$  is closed in  $\theta$ .

In the following, we inductively construct a sequence  $\langle \langle C_{\gamma+\omega \cdot \alpha} \mid \alpha < \lambda \rangle \mid \gamma \in A \rangle$  that satisfies the following statements for all  $\gamma \in A$ :

- (i)  $C_\gamma = D_\gamma$ .
- (ii) If  $0 < \alpha < \lambda$ , then  $C_{\gamma+\omega \cdot \alpha}$  is a club in  $\gamma + \omega \cdot \alpha$  with  $C_{\gamma+\omega \cdot \alpha} \cap S = \emptyset$ .
- (iii) If  $\tilde{\gamma} < \gamma$  and  $0 < \alpha < \lambda$ , then there is  $\tilde{\gamma} < \min(A \setminus \tilde{\gamma}) + \omega \cdot \lambda$  with

$$C_{\gamma+\omega \cdot \alpha} \cap \tilde{\gamma} = (C_{\tilde{\gamma}} \cup \{\tilde{\gamma}\}) \cap \tilde{\gamma}. \quad (1)$$

- (iv) If  $\gamma \in A \setminus S$  and  $\tilde{\gamma} \in \text{Lim}(A \cap D_\gamma)$ , then  $C_{\gamma+\omega \cdot \alpha}$  is an end-extension of  $C_{\tilde{\gamma}+\omega \cdot \alpha}$  for all  $0 < \alpha < \lambda$ .
- (v) If  $\tilde{\gamma} \in A \cap \gamma$ , then there is  $\tilde{\lambda} < \lambda$  such that  $C_{\gamma+\omega \cdot \alpha}$  is an end-extension of  $C_{\tilde{\gamma}+\omega \cdot \alpha}$  for all  $\tilde{\lambda} < \alpha < \lambda$ .

In the construction of this sequence, we will distinguish several cases.

**Case 0:**  $\gamma = 0$ . Define  $C_{\omega \cdot \alpha} = \omega \cdot \alpha$  for all  $\alpha < \lambda$ .

**Case 1:**  $A$  is bounded in  $\gamma > 0$ . Pick ordinals  $\gamma_0 \leq \gamma_1 < \gamma$  with the property that  $\gamma_1 = \max(A \cap \gamma)$  and either  $\text{Lim}(A \cap D_\gamma) \neq \emptyset$  and  $\gamma_0 = \max(\text{Lim}(A \cap D_\gamma))$  or  $\text{Lim}(A \cap D_\gamma) = \emptyset$  and  $\gamma_0 = \gamma_1$ . By our induction hypothesis, there is  $\lambda_1 < \lambda$  such that  $C_{\gamma_1+\omega \cdot \alpha}$  is an end-extension of  $C_{\gamma_0+\omega \cdot \alpha}$  for all  $\lambda_1 < \alpha < \lambda$ . Set  $\lambda_0 = 0$  and  $\lambda_2 = \lambda$ . Given  $0 < \alpha < \lambda$ , pick  $i < 2$  with  $\lambda_i < \alpha \leq \lambda_{i+1}$  and define

$$C_{\gamma+\omega \cdot \alpha} = C_{\gamma_i+\omega \cdot \alpha} \cup \{\gamma_i + \omega \cdot \alpha\} \cup (\gamma, \gamma + \omega \cdot \alpha).$$

Given  $0 < \alpha < \lambda$ , the set  $C_{\gamma+\omega \cdot \alpha}$  is a club in  $\gamma + \omega \cdot \alpha$  that is disjoint from  $S$ . Moreover, for all  $\tilde{\gamma} < \gamma$  and  $0 < \alpha < \lambda$ , we can find  $i < 2$  with

$$C_{\gamma+\omega \cdot \alpha} \cap \tilde{\gamma} = (C_{\gamma_i+\omega \cdot \alpha} \cup \{\gamma_i + \omega \cdot \alpha\}) \cap \tilde{\gamma}.$$

Together with our induction hypothesis, this allows us to find  $\tilde{\gamma} < \min(A \setminus \tilde{\gamma}) + \omega \cdot \lambda$  such that (1) holds. If  $\text{Lim}(A \cap D_\gamma) \neq \emptyset$ , then our induction hypothesis implies that  $C_{\gamma+\omega \cdot \alpha}$  is an end-extension of  $C_{\tilde{\gamma}+\omega \cdot \alpha}$  for all  $\tilde{\gamma} \in \text{Lim}(A \cap D_\gamma)$  and  $0 < \alpha < \lambda$ , because the above construction ensures that  $C_{\gamma+\omega \cdot \alpha}$  is an end-extension of  $C_{\gamma_0+\omega \cdot \alpha}$  for all  $0 < \alpha < \lambda$ . Finally, for every  $\tilde{\gamma} \in A \cap \gamma$  there is  $\lambda_1 \leq \tilde{\lambda} < \lambda$  such that  $C_{\gamma_1+\omega \cdot \alpha}$  is an end-extension of  $C_{\tilde{\gamma}+\omega \cdot \alpha}$  for all  $\tilde{\lambda} < \alpha < \lambda$  and this implies that  $C_{\gamma+\omega \cdot \alpha}$  is an end-extension of  $C_{\tilde{\gamma}+\omega \cdot \alpha}$  for all  $\tilde{\lambda} < \alpha < \lambda$ .

**Case 2:**  $A$  is unbounded in  $\gamma$  and  $\text{Lim}(A \cap D_\gamma)$  is bounded in  $\gamma$ . Then  $\text{cof}(\gamma) = \omega$  and we can pick a strictly increasing sequence  $\langle \gamma_n \in A \mid n < \omega \rangle$  that is cofinal in  $\gamma$  such that  $\text{Lim}(A \cap D_\gamma) \neq \emptyset$  implies  $\gamma_0 = \max(\text{Lim}(A \cap D_\gamma))$ . By our induction hypothesis, there is a strictly increasing sequence  $\langle \lambda_n < \lambda \mid n < \omega \rangle$  such that  $\lambda_0 = 0$ ,  $C_{\gamma_{n+1}+\omega \cdot \alpha}$  is an end-extension of  $C_{\gamma_n+\omega \cdot \alpha}$  for all  $n < \omega$  and  $\lambda_{n+1} \leq \alpha < \lambda$  and, if  $\text{cof}(\lambda) = \omega$ , then this sequence is cofinal in  $\lambda$ . Then  $C_{\gamma_n+\omega \cdot \alpha}$  is an end-extension of  $C_{\gamma_m+\omega \cdot \alpha}$  for all  $m \leq n < \omega$  and  $\lambda_n \leq \alpha < \lambda$ . Set  $\lambda_\omega = \sup_{n < \omega} \lambda_n \leq \lambda$ . Note that  $\lambda_\omega < \lambda$  implies that  $\lambda > \omega$  and  $\gamma \notin S$ . Given  $0 < \alpha < \lambda_\omega$ , let  $n < \omega$  be maximal with  $\lambda_n \leq \alpha$  and define

$$C_{\gamma+\omega \cdot \alpha} = C_{\gamma_n+\omega \cdot \alpha} \cup \{\gamma_n + \omega \cdot \alpha\} \cup (\gamma, \gamma + \omega \cdot \alpha).$$

Given  $\lambda_\omega \leq \alpha < \lambda$ , define

$$C_{\gamma+\omega \cdot \alpha} = \bigcup \{C_{\gamma_n+\omega \cdot \alpha} \mid n < \omega\} \cup [\gamma, \gamma + \omega \cdot \alpha).$$

Then the set  $C_{\gamma+\omega\cdot\alpha}$  is a club in  $\gamma+\omega\cdot\alpha$  that it is disjoint from  $S$  for all  $0 < \alpha < \lambda$ . If  $\tilde{\gamma} < \gamma$  and  $0 < \alpha < \lambda$ , then there is an  $n < \omega$  with

$$C_{\gamma+\omega\cdot\alpha} \cap \tilde{\gamma} = (C_{\gamma_n+\omega\cdot\alpha} \cup \{\gamma_n + \omega\cdot\alpha\}) \cap \tilde{\gamma}.$$

In combination with our induction hypothesis, this yields a  $\bar{\gamma} < \min(A \setminus \tilde{\gamma}) + \omega\cdot\lambda$  such that (1) holds. Next, observe that  $\text{Lim}(A \cap D_\gamma) \neq \emptyset$  implies that  $C_{\gamma+\omega\cdot\alpha}$  is an end-extension of  $C_{\gamma_0+\omega\cdot\alpha}$  for all  $0 < \alpha < \lambda$ . If  $\bar{\gamma} \in \text{Lim}(A \cap D_\gamma)$  and  $0 < \alpha < \lambda$ , then  $\gamma_0 \in A \setminus S$ ,  $\bar{\gamma} \in \text{Lim}(A \cap D_{\gamma_0})$  and our induction hypothesis together with the above observation shows that  $C_{\gamma+\omega\cdot\alpha}$  is an end-extension of  $C_{\bar{\gamma}+\omega\cdot\alpha}$ . Finally, pick  $\tilde{\gamma} \in A \cap \gamma$ . By our induction hypothesis, there are  $n < \omega$  and  $\lambda_n \leq \tilde{\lambda} < \lambda$  such that  $\tilde{\gamma} \leq \gamma_n$  and  $C_{\gamma_n+\omega\cdot\alpha}$  is an end-extension of  $C_{\tilde{\gamma}+\omega\cdot\alpha}$  for all  $\tilde{\lambda} < \alpha < \lambda$ . Fix  $\tilde{\lambda} < \alpha < \lambda$  and let  $\beta \leq \omega$  be maximal with  $\lambda_\beta \leq \alpha$ . Then  $\beta \geq n$ ,  $C_{\gamma+\omega\cdot\alpha}$  is an end-extension of  $C_{\gamma_n+\omega\cdot\alpha}$  and therefore  $C_{\gamma+\omega\cdot\alpha}$  is an end-extension of  $C_{\tilde{\gamma}+\omega\cdot\alpha}$ .

**Case 3:**  $\text{Lim}(A \cap D_\gamma)$  is unbounded in  $\gamma$  and  $\gamma \notin S$ . Define

$$C_{\gamma+\omega\cdot\alpha} = \bigcup \{C_{\bar{\gamma}+\omega\cdot\alpha} \mid \bar{\gamma} \in \text{Lim}(A \cap D_\gamma)\} \cup [\gamma, \gamma + \omega\cdot\alpha)$$

for all  $0 < \alpha < \lambda$ . If  $\gamma_0, \gamma_1 \in \text{Lim}(A \cap D_\gamma)$  with  $\gamma_0 < \gamma_1$ , then  $\gamma_0 \in \text{Lim}(A \cap D_{\gamma_1})$  and our induction hypothesis implies that  $C_{\gamma_1+\omega\cdot\alpha}$  is an end-extension of  $C_{\gamma_0+\omega\cdot\alpha}$  for all  $0 < \alpha < \lambda$ . This shows that for all  $0 < \alpha < \lambda$ , the set  $C_{\gamma+\omega\cdot\alpha}$  is a club in  $\gamma + \omega\cdot\alpha$  that is disjoint from  $S$ . Moreover, if  $\tilde{\gamma} < \gamma$  and  $0 < \alpha < \lambda$ , then there is  $\gamma_0 \in \text{Lim}(A \cap D_\gamma)$  with  $C_{\gamma+\omega\cdot\alpha} \cap \tilde{\gamma} = C_{\gamma_0+\omega\cdot\alpha} \cap \tilde{\gamma}$  and, by our induction hypothesis, this shows that there is a  $\bar{\gamma} < \min(A \setminus \tilde{\gamma}) + \omega\cdot\lambda$  such that (1) holds. Next, our induction hypothesis shows that  $C_{\gamma+\omega\cdot\alpha}$  is an end-extension of  $C_{\bar{\gamma}+\omega\cdot\alpha}$  for all  $\bar{\gamma} \in \text{Lim}(A \cap D_\gamma)$  and  $0 < \alpha < \lambda$ . Finally, if  $\tilde{\gamma} \in A \cap \gamma$ , then our induction hypothesis implies that there is  $\bar{\gamma} \in \text{Lim}(A \cap D_\gamma)$  and  $\tilde{\lambda} < \lambda$  such that  $\tilde{\gamma} \leq \bar{\gamma}$ ,  $C_{\bar{\gamma}+\omega\cdot\alpha}$  is an end-extension of  $C_{\tilde{\gamma}+\omega\cdot\alpha}$  for all  $\tilde{\lambda} < \alpha < \lambda$  and  $C_{\gamma+\omega\cdot\alpha}$  is an end-extension of  $C_{\tilde{\gamma}+\omega\cdot\alpha}$  for all  $\tilde{\lambda} < \alpha < \lambda$ .

**Case 4:**  $\text{Lim}(A \cap D_\gamma)$  is unbounded in  $\gamma$  and  $\gamma \in S$ . Then  $\text{cof}(\gamma) = \lambda$  and we can pick a strictly increasing continuous sequence  $\langle \gamma_\alpha \in \text{Lim}(A \cap D_\gamma) \mid \alpha < \lambda \rangle$  that is cofinal in  $\gamma$ . Given  $0 < \alpha < \lambda$ , define

$$C_{\gamma+\omega\cdot\alpha} = C_{\gamma_\alpha+\omega\cdot\alpha} \cup \{\gamma_\alpha + \omega\cdot\alpha\} \cup (\gamma, \gamma + \omega\cdot\alpha).$$

Given  $0 < \alpha < \lambda$ , the set  $C_{\gamma+\omega\cdot\alpha}$  is a club in  $\gamma + \omega\cdot\alpha$  that it is disjoint from  $S$ . Next, if  $\tilde{\gamma} < \gamma$  and  $0 < \alpha < \lambda$ , then

$$C_{\gamma+\omega\cdot\alpha} \cap \tilde{\gamma} = (C_{\gamma_\alpha+\omega\cdot\alpha} \cup \{\gamma_\alpha + \omega\cdot\alpha\}) \cap \tilde{\gamma}$$

and our induction hypothesis yields  $\bar{\gamma} < \min(A \setminus \tilde{\gamma}) + \omega\cdot\lambda$  such that (1) holds. Finally, pick  $\tilde{\gamma} \in A \cap \gamma$ . Then our induction hypothesis allows us to find  $\tilde{\alpha} \leq \tilde{\lambda} < \lambda$  such that  $\tilde{\gamma} \leq \gamma_{\tilde{\alpha}}$  and  $C_{\gamma_{\tilde{\alpha}}+\omega\cdot\alpha}$  is an end-extension of  $C_{\tilde{\gamma}+\omega\cdot\alpha}$  for all  $\tilde{\lambda} < \alpha < \lambda$ . Fix  $\tilde{\lambda} < \alpha < \lambda$ . Since  $\gamma_\alpha \in A \setminus S$  and  $\gamma_{\tilde{\alpha}} \in \text{Lim}(A \cap D_{\gamma_\alpha})$ , our induction hypothesis implies that  $C_{\gamma_\alpha+\omega\cdot\alpha}$  is an end-extension of  $C_{\gamma_{\tilde{\alpha}}+\omega\cdot\alpha}$ . This allows us to conclude that  $C_{\gamma+\omega\cdot\alpha}$  is an end-extension of  $C_{\tilde{\gamma}+\omega\cdot\alpha}$ .

This completes our inductive construction. We let  $\vec{C}$  denote the resulting  $C$ -sequence. Set  $\mathbb{T} = \mathbb{T}(\rho_0^{\vec{C}})$  and  $\rho_0 = \rho_0^{\vec{C}}$ .

**Claim.** *The tree  $\mathbb{T}$  is a  $\theta$ -Aronszajn tree.*

*Proof of the Claim.* Pick  $\gamma < \theta$ . Since  $\vec{D}$  is a  $\square(\theta)$ -sequence, we have

$$|\{C_\delta \cap \gamma \mid \delta \in A \setminus \gamma\}| \leq |\gamma| + \aleph_0 < \theta.$$

Moreover, the above construction ensures that

$$\begin{aligned} & |\{C_{\delta+\omega \cdot \alpha} \cap \gamma \mid \gamma + \omega \cdot \lambda \leq \delta \in A, 0 < \alpha < \lambda\}| \\ & \leq |\{(C_\delta \cup \{\delta\}) \cap \gamma \mid \delta < \min(A \setminus \gamma) + \omega \cdot \lambda\}| < \theta. \end{aligned}$$

In combination, this shows that  $|\{C_\delta \cap \gamma \mid \gamma \leq \delta < \theta\}| < \theta$  holds.

Together with Lemma 4.2, the above computations show that  $|\mathbb{T}(\gamma)| < \theta$  holds for all  $\gamma < \theta$ . Since  $\vec{C}$  avoids  $S$ , we can apply Proposition 1.2 and Lemma 4.9 to conclude that there are no cofinal branches through  $\mathbb{T}$ . This shows that  $\mathbb{T}$  is a  $\theta$ -Aronszajn tree.  $\square$

Given  $\gamma \in A$  and  $\alpha < \lambda$ , define

$$t_\gamma(\alpha) = \rho_0(\cdot, \gamma + \omega \cdot (1 + \alpha)) \upharpoonright \gamma \in \mathbb{T}(\gamma).$$

**Claim.** *If  $\gamma, \tilde{\gamma} \in A$  with  $\tilde{\gamma} < \gamma$ , then there is  $\tilde{\lambda} < \lambda$  such that  $t_{\tilde{\gamma}}(\alpha) <_{\mathbb{T}} t_\gamma(\alpha)$  for all  $\tilde{\lambda} < \alpha < \lambda$ .*

*Proof of the Claim.* By the above construction, we can find  $\tilde{\lambda} < \lambda$  with the property that  $C_{\gamma+\omega \cdot \alpha}$  is an end-extension of  $C_{\tilde{\gamma}+\omega \cdot \alpha}$  for all  $\tilde{\lambda} < \alpha < \lambda$ . By the definition of  $\rho_0$ , this implies that  $t_{\tilde{\gamma}}(\alpha) <_{\mathbb{T}} t_\gamma(\alpha)$  holds for all  $\tilde{\lambda} < \alpha < \lambda$ .  $\square$

Given  $\gamma < \theta$  and  $\alpha < \lambda$ , set  $\delta = \min(A \setminus \gamma)$  and define  $b_\gamma(\alpha)$  to be the unique element of  $\mathbb{T}(\gamma)$  with  $b_\gamma(\alpha) \leq_{\mathbb{T}} t_\delta(\alpha)$ . Then the above claim shows that the resulting sequence  $\langle b_\gamma : \lambda \longrightarrow \mathbb{T}(\gamma) \mid \gamma < \theta \rangle$  is a  $\lambda$ -ascent path through  $\mathbb{T}$ .  $\square$

Before we prove Theorem 1.12, we use the above theorem to reprove a result of [25] on lower bounds for the consistency strength of the conclusion of Theorem 1.13,(iii). Note that the derived lower bound is strictly smaller than the upper bound given by the theorem (see Question 6.6).

**Corollary 4.13.** *Assume that  $\theta > \omega_1$  and there is a  $\square(\theta)$ -sequence  $\vec{C}$  with the property that the tree  $\mathbb{T}(\rho_0^{\vec{C}})$  is special.*

- (i) *If  $\theta$  is not a successor of a cardinal of cofinality  $\omega$ , then there is a  $C$ -sequence  $\vec{C}$  of length  $\theta$  with the property that  $\mathbb{T}(\rho_0^{\vec{C}})$  is a  $\theta$ -Aronszajn tree that contains an  $\omega$ -ascent path.*
- (ii) *If  $\theta$  is a successor of a cardinal of cofinality  $\omega$ , then there is a  $C$ -sequence  $\vec{C}$  of length  $\theta$  with the property that  $\mathbb{T}(\rho_0^{\vec{C}})$  is a  $\theta$ -Aronszajn tree that contains an  $\omega_1$ -ascent path.*

*Proof.* Let  $\kappa < \theta$  be a regular cardinal with the property that  $\theta$  is not a successor of a cardinal of cofinality  $\kappa$ . Then a combination of [25, Lemma 4] with [11, Proposition 30] implies that there is a  $\square(\theta)$ -sequence  $\vec{D}$  and  $S \subseteq S_\kappa^\theta$  stationary in  $\theta$  such that  $\vec{D}$  avoids  $S$ . By Theorem 4.12, this implies the conclusion of the corollary.  $\square$

Note that results of Jensen show that the assumption of Corollary 4.13 holds if  $\theta$  is not a Mahlo cardinal in  $L$  (see [25, Theorem 2]). Moreover, Jensen's classical definition of  $\square_\kappa$ -sequences provides examples for sequences with the above property.

**Definition 4.14** (Jensen). Let  $\kappa$  be an infinite cardinal. A  $\square_\kappa$ -sequence is a  $\kappa^+$ -coherent  $C$ -sequence  $\vec{C} = \langle C_\gamma \mid \gamma < \kappa^+ \rangle$  with  $\text{otp}(C_\gamma) \leq \kappa$  for all  $\gamma < \kappa^+$ .

Basic computations contained in [25] show that for every  $\square_\kappa$ -sequence  $\vec{C}$ , the tree  $\mathbb{T}(\rho_0^{\vec{C}})$  is a special  $\kappa^+$ -Aronszajn tree. Using results from inner model theory, this shows that the non-existence of Aronszajn trees containing ascending paths of width at most  $\omega_1$  at the successor of a singular cardinal or the successor of a weakly compact has very large consistency strength.

*Proof of Theorem 1.12.* Assume that  $V$  is a Jensen-style extender model and let  $\vartheta$  be an uncountable regular cardinal.

First, assume that  $\vartheta$  is weakly compact. Let  $\nu < \vartheta$ , let  $\mathbb{P} = \prod_{\eta < \rho} \mathbb{P}_\eta$  be a  $\nu$ -support product of  $\vartheta$ -Knaster partial orders and let  $\langle \vec{p}_\gamma \mid \gamma < \vartheta \rangle$  be a sequence of conditions in  $\mathbb{P}$ . By the  $\Delta$ -system lemma, there is an unbounded subset  $S$  of  $\vartheta$  and a subset  $R$  of  $\rho$  of cardinality at most  $\nu$  such that  $\text{supp}(p_\eta) \cap \text{supp}(p_{\bar{\eta}}) = R$  for all  $\eta, \bar{\eta} \in S$  with  $\eta \neq \bar{\eta}$ . Let  $c : S \times S \rightarrow R \cup \{\rho\}$  denote the unique function with the property that the following statements hold for all  $\gamma, \bar{\gamma} \in S$ :

- (i)  $c(\gamma, \bar{\gamma}) = \rho$  if and only if the conditions  $\vec{p}_\gamma$  and  $\vec{p}_{\bar{\gamma}}$  are compatible in  $\mathbb{P}$ .
- (ii) If  $c(\gamma, \bar{\gamma}) < \rho$ , then  $c(\gamma, \bar{\gamma})$  is the minimal element  $\eta$  of  $R$  with the property that the conditions  $\vec{p}_\gamma(\eta)$  and  $\vec{p}_{\bar{\gamma}}(\eta)$  are incompatible in  $\mathbb{P}_\eta$ .

Since  $\vartheta$  is weakly compact and each partial order  $\mathbb{P}_\eta$  satisfies the  $\vartheta$ -chain condition, there is  $U \subseteq S$  unbounded in  $\vartheta$  with  $c(\gamma, \bar{\gamma}) = \rho$  for all  $\gamma, \bar{\gamma} \in U$ . Then the sequence  $\langle \vec{p}_\gamma \mid \gamma \in U \rangle$  consists of pairwise compatible conditions in  $\mathbb{P}$ .

Next, assume that  $\vartheta$  is neither weakly compact nor the successor of a subcompact cardinal. If  $\vartheta$  is inaccessible, then [30, Theorem 0.1] shows that there is a  $\square(\vartheta)$ -sequence  $\vec{C}$  and  $S \subseteq S_\omega^\vartheta$  stationary in  $\vartheta$  such that  $\vec{C}$  avoids  $S$ . In the other case, if  $\vartheta$  is the successor of a cardinal  $\kappa$  that is not subcompact, then [17, Theorem 0.1] shows that there is a  $\square_\kappa$ -sequences and a combination of [11, Proposition 30] with [11, Corollary 32] shows that in this case we can also find a  $\square(\vartheta)$ -sequence  $\vec{C}$  and  $S \subseteq S_\omega^\vartheta$  stationary in  $\vartheta$  such that  $\vec{C}$  avoids  $S$ . By Lemma 4.9 and Theorem 4.12, this implies that there is a club  $D$  in  $\vartheta$  and a  $\vartheta$ -Aronszajn tree  $\mathbb{T}$  such that there is an ascending path of width  $\omega$  through  $\mathbb{T}$  and the set  $D \cap S$  is non-stationary with respect to  $\mathbb{T}$ . Then Lemma 4.10 shows that the partial order  $\mathbb{P}_\omega(\mathbb{T})$  is  $\vartheta$ -Knaster and Proposition 2.3 implies that the full support product  $\prod_\omega \mathbb{P}_\omega(\mathbb{T})$  is not  $\vartheta$ -Knaster.

Finally, assume that  $\vartheta$  is the successor of a subcompact cardinal  $\kappa$ . Since subcompact cardinals are weakly compact, we have  $\kappa = \kappa^{<\kappa}$  and a classical result of Specker shows that there is a normal special  $\vartheta$ -Aronszajn tree  $\mathbb{T}$ . Then Lemma 4.10 implies that the partial order  $\mathbb{P}_\omega(\mathbb{T})$  is  $\vartheta$ -Knaster. In this situation, Proposition 1.5 shows that  $\mathbb{T}$  contains an ascending path of width  $\kappa$  and Proposition 2.3 shows that the full support product  $\prod_\kappa \mathbb{P}_\omega(\mathbb{T})$  is not  $\vartheta$ -Knaster.  $\square$

## 5. PFA AND ASCENDING PATHS

In this section, we prove the three statements of Theorem 1.14. We start by showing how the theory of *guessing models* developed by Viale and Weiss in [28] can be used to show that the Proper Forcing Axiom implies the non-existence of trees containing ascending paths of countable width without cofinal branches. Basically the same implication was independently proven by Lambie-Hanson in [12, Section 10]. For sake of completeness, we still present this application of the results of [28]. Given a set  $X$  and a cardinal  $\kappa$ , we let  $\mathcal{P}_\kappa(X)$  denote the collection of all subsets of  $X$  of cardinality less than  $\kappa$ .

**Definition 5.1.** Let  $\vartheta$  be an uncountable regular cardinal and let  $M$  be an elementary substructure  $H(\vartheta)$ .

- (i) A set  $X \in H(\vartheta)$  is *M-approximated* if  $X \cap Y \in M$  for all  $Y \in M \cap \mathcal{P}_{\omega_1}(M)$ .
- (ii) A set  $X \in H(\vartheta)$  is *M-guessed* if there is an  $Y \in M$  with  $X \cap M = Y \cap M$ .
- (iii)  $M$  is a *guessing model* if for all  $X \in M$  and every  $Y \subseteq X$  that is *M-approximated*, the set  $X$  is *M-guessed*.

**Definition 5.2** ([28]). Given an uncountable regular cardinal  $\kappa$ , we let  $\text{ISP}(\kappa)$  denote the statement that for every regular cardinal  $\vartheta > \kappa$ , the collection of all guessing models is stationary in  $M \in \mathcal{P}_\kappa(H(\vartheta))$ .

**Theorem 5.3** ([28, Theorem 4.8]). PFA implies  $\text{ISP}(\omega_2)$ .

The following results also appeared as [12, Theorem 10.2].

**Lemma 5.4.** *If  $\text{ISP}(\kappa)$  holds and  $\theta \geq \omega_2$ , then every narrow  $\theta$ -system of width  $\omega$  has a cofinal branch.*

*Proof.* Let  $\mathcal{R}$  be a narrow  $\theta$ -system of width  $\omega$ , let  $\vartheta > \theta$  be a regular cardinal with  $\mathcal{R} \in H(\vartheta)$  and let  $M \in \mathcal{P}_{\omega_2}(H(\vartheta))$  be a guessing model with  $\mathcal{R} \in M$ . Set  $\delta = \sup(M \cap \theta) < \theta$ . Then  $\text{cof}(\delta) = \omega_1$ , because our assumptions imply that every subset of  $M \cap \delta$  of order-type  $\omega$  is contained in  $M$ . By our assumptions, there are  $\beta < \lambda \subseteq M$  and  $R \in \mathcal{R} \subseteq M$  such that the set

$$\{\gamma \in D \cap M \mid \exists \alpha < \lambda \langle \gamma, \alpha \rangle R \langle \delta, \beta \rangle\}$$

is unbounded in  $\delta$ . Set

$$A = \{\langle \gamma, \alpha \rangle \in D \times \lambda \mid \langle \gamma, \alpha \rangle R \langle \delta, \beta \rangle\}.$$

Then  $A \subseteq D \times \lambda \in M$  is a branch through  $\mathcal{R}$ .

Pick  $X \in M \cap \mathcal{P}_{\omega_1}(M)$ . Since  $\text{cof}(\delta) = \omega_1$ , we can find  $\gamma \in D \cap M$  and  $\alpha < \lambda$  such that  $A \cap X \subseteq \gamma \times \lambda$  and  $\langle \gamma, \alpha \rangle R \langle \delta, \beta \rangle$ . Then

$$A \cap X = \{\langle \bar{\gamma}, \bar{\alpha} \rangle \in D \times \lambda \mid \langle \bar{\gamma}, \bar{\alpha} \rangle \in X, \langle \bar{\gamma}, \bar{\alpha} \rangle R \langle \gamma, \alpha \rangle\} \in M.$$

The above computations show that  $A$  is *M-approximated* and hence it is *M-guessed*. This shows that there is  $B \in M$  with  $B \subseteq D \times \lambda$  and  $A \cap M = B \cap M$ . In  $M$ , the set  $B$  is a cofinal  $R$ -branch though  $\mathcal{R}$ . By elementarity,  $\mathcal{R}$  has a cofinal branch in  $V$ .  $\square$

In the following, we will show that PFA is compatible with the existence of an  $\omega_3$ -Souslin tree that contains an ascending path of width  $\omega_1$ . This proof uses the weak square principle introduced by Baumgartner in unpublished work and parallels the constructions of [12, Section 9].

**Definition 5.5** (Baumgartner). Let  $\kappa$  be an infinite regular cardinal, let  $B$  be a subset of  $\kappa^+$  with  $S_\kappa^{\kappa^+} \subseteq B \subseteq \text{Lim}$  and let  $\vec{C} = \langle C_\gamma \mid \gamma < \kappa^+ \rangle$  be a  $C$ -sequence. We say that  $\vec{C}$  is a  $\square_\kappa^B$ -sequence if the following statements hold:

- (i)  $\text{otp}(C_\gamma) \leq \kappa$  for all  $\gamma < \kappa^+$ .
- (ii) If  $\gamma \in B \cap \text{Lim}$  and  $\bar{\gamma} \in \text{Lim}(C_\gamma)$ , then  $\bar{\gamma} \in B$  and  $C_{\bar{\gamma}} = C_\gamma \cap \bar{\gamma}$ .

In contrast to  $\square_\kappa$ -sequences, these sequences can be added by  $< \kappa$ -directed closed forcings that preserve the regularity of  $\kappa^+$ .

**Lemma 5.6** ([2, Fact 2.7]). *If  $\kappa$  is an infinite regular cardinal, then there is a partial order  $\mathbb{P}$  with the following properties:*

- (i)  $\mathbb{P}$  is  $<\kappa$ -directed closed and  $(\kappa + 1)$ -strategically closed.
- (ii) If  $G$  is  $\mathbb{P}$ -generic over  $V$ , then there is a  $\square_\kappa^B$ -sequence in  $V[G]$ .

The following observation shows that we can modify a given  $\square_\kappa^B$ -sequence to obtain a sequence that avoids a stationary subset of a given stationary set.

**Proposition 5.7.** *Let  $\kappa$  be an infinite regular cardinal with the property that there is a  $\square_\kappa^B$ -sequence.*

- (i) *The set  $B$  is a fat stationary subset of  $\kappa^+$ . In particular,  $S \cap S_\nu^{\kappa^+}$  is stationary in  $\kappa^+$  for every regular cardinal  $\nu \leq \kappa$ .*
- (ii) *If  $S \subseteq B$  is stationary in  $\kappa^+$ , then there is  $A \subseteq S$  stationary in  $\kappa^+$  and a  $\square_\kappa^B$ -sequence  $\vec{C} = \langle C_\gamma \mid \gamma < \kappa^+ \rangle$   $A \cap \text{Lim}(C_\gamma) = \emptyset$  holds for all  $\gamma \in B$ .*

*Proof.* Let  $\vec{D} = \langle D_\gamma \mid \gamma < \kappa^+ \rangle$  be a  $\square_\kappa^B$ -sequence.

(i) We may assume that  $\kappa$  is uncountable, because otherwise the statement holds trivially. Let  $C$  be a club in  $\kappa^+$ . Pick  $\gamma \in \text{Lim}(C) \cap S_\kappa^{\kappa^+} \subseteq B$ , then the set  $(C \cap \text{Lim}(C_\gamma)) \cup \{\gamma\}$  is a closed subset of  $B \cap C$  of order-type  $\kappa + 1$ . By [1, Lemma 1.2], this argument shows that  $B$  is fat stationary in  $\kappa^+$ .

(ii) By our assumptions, we have  $\text{otp}(C_\gamma) < \gamma$  for all  $\gamma \in S \setminus (\kappa + 1)$ . This allows us to find  $\xi \leq \kappa$  and  $A \subseteq S$  stationary in  $\kappa^+$  such that  $\text{otp}(D_\gamma) = \xi$  for all  $\gamma \in A$ . Then  $|A \cap \text{Lim}(D_\gamma)| \leq 1$  holds for all  $\gamma \in B$ . If  $\gamma \in B$  and  $A \cap \text{Lim}(D_\gamma) = \{\bar{\gamma}\}$ , then we define  $C_\gamma = D_\gamma \cap (\bar{\gamma}, \gamma)$ . In the other case, if either  $\gamma \in \kappa^+ \setminus B$  or  $\gamma \in B$  and  $A \cap \text{Lim}(D_\gamma) = \emptyset$ , then we define  $C_\gamma = D_\gamma$ . The resulting sequence  $\langle C_\gamma \mid \gamma < \kappa^+ \rangle$  is a  $C$ -sequence with  $\text{otp}(C_\gamma) \leq \kappa$  and  $A \cap \text{Lim}(C_\gamma) = \emptyset$  for all  $\gamma < \kappa^+$ . If  $\gamma \in B$  and  $\bar{\gamma} \in \text{Lim}(C_\gamma)$ , then  $\bar{\gamma} \in \text{Lim}(D_\gamma)$ ,  $\bar{\gamma} \in B \setminus A$ ,  $D_{\bar{\gamma}} = D_\gamma \cap \bar{\gamma}$  and the above computation implies that  $C_{\bar{\gamma}} = C_\gamma \cap \bar{\gamma}$ .  $\square$

In combination with certain fragments of the GCH, the above principle allows us to construct Souslin trees containing ascending paths of small width.

**Theorem 5.8.** *Let  $\kappa$  be an uncountable regular cardinal that satisfies  $2^\kappa = \kappa^+$  and  $2^{\kappa^+} = \kappa^{++}$ . If there is a  $\square_{\kappa^+}^B$ -sequence, then there is a  $\kappa^{++}$ -Souslin tree that contains a  $\kappa$ -ascent path.*

*Proof of the Theorem.* Set  $\vartheta = \kappa^{++}$ . By our assumptions and Proposition 5.7, there is a  $\square_\kappa^B$ -sequence  $\vec{C} = \langle C_\gamma \mid \gamma < \vartheta \rangle$  and  $A \subseteq B \cap S_\kappa^\vartheta$  stationary in  $\vartheta$  such that  $A \cap \text{Lim}(C_\gamma) = \emptyset$  for all  $\gamma \in B$ . Set  $\bar{B} = B \cup \{0\}$ . Results of Shelah (see [19]) show that our assumptions imply that  $\diamond(E)$  holds for all  $E \subseteq S_{\leq \kappa}^\vartheta$  stationary in  $\vartheta$ . This allows us to fix a  $\diamond(A)$ -sequence  $\langle A_\gamma \mid \gamma \in A \rangle$ .

We construct the following objects by induction on  $\gamma < \vartheta$ :

- (i) A subtree  $\mathbb{T}$  of  ${}^{<\vartheta}2$  of height  $\vartheta$  (i.e.  $\mathbb{T}$  is a tree of height  $\vartheta$  that consists of functions  $t : \gamma \rightarrow 2$  with  $\gamma < \vartheta$  ordered by inclusion and is closed under initial segments) with the following properties:
  - (a) If  $\gamma < \vartheta$ , then  $|\mathbb{T}(\gamma)| < \vartheta$  and every element of  $\mathbb{T}(\gamma)$  has two distinct direct successors in  $\mathbb{T}(\gamma + 1)$ .
  - (b) If  $\nu \leq \kappa$  and  $\langle t_\xi \mid \xi < \nu \rangle$  is an ascending sequence in  $\mathbb{T}$  with the property that  $\sup_{\xi < \nu} \text{lh}_{\mathbb{T}}(t_\xi) \notin A$ , then  $\bigcup_{\xi < \nu} t_\xi \in \mathbb{T}$ .
  - (c) If  $t \in \mathbb{T}$  and  $\text{lh}_{\mathbb{T}}(t) \leq \gamma < \vartheta$ , then there is  $u \in \mathbb{T}(\gamma)$  with  $t \subseteq u$ .
- (ii) An injection  $\iota : \mathbb{T} \rightarrow \vartheta$  with the following properties:
  - (a) If  $s, t \in \mathbb{T}$  with  $\text{lh}_{\mathbb{T}}(s) < \text{lh}_{\mathbb{T}}(t)$ , then  $\iota(s) < \iota(t)$ .

- (b) If  $\gamma \in A$  and the  $(\iota \upharpoonright \mathbb{T}_{<\gamma})$ -preimage of  $A_\gamma$  is a maximal antichain in  $\mathbb{T}_{<\gamma}$ , then for every  $u \in \mathbb{T}(\gamma)$ , there is a  $t \in \mathbb{T}_{<\gamma}$  with  $t \subseteq u$  and  $\iota(t) \in A_\gamma$ .
- (iii) A sequence  $\langle a_\gamma : \kappa \longrightarrow \mathbb{T}(\gamma) \mid \gamma \in \bar{B} \rangle$  of functions such that the following statements hold:
  - (a) If  $\gamma \in B \setminus A$  and  $\bar{\gamma} \in \text{Lim}(C_\gamma)$ , then  $a_{\bar{\gamma}}(\alpha) \subseteq a_\gamma(\alpha)$  for all  $\alpha < \kappa$ .
  - (b) If  $\gamma, \bar{\gamma} \in \bar{B}$  with  $\bar{\gamma} < \gamma$ , then there is a  $\bar{\kappa} < \kappa$  with  $a_{\bar{\gamma}}(\alpha) \subseteq a_\gamma(\alpha)$  for all  $\bar{\kappa} \leq \alpha < \kappa$ .

In the inductive construction of the above objects, we have to distinguish several cases and subclasses:

**Case 0:**  $\gamma = 0$ . Set  $\mathbb{T}(0) = \{\emptyset\}$  and  $a_0(\alpha) = \emptyset$  for all  $\alpha < \kappa$ .

**Case 1:**  $\gamma = \bar{\gamma} + 1$ . Define  $\mathbb{T}(\gamma) = \{t \in {}^\gamma 2 \mid t \upharpoonright \bar{\gamma} \in \mathbb{T}(\bar{\gamma})\}$ . In this situation, our induction hypothesis ensures that  $|\mathbb{T}(\gamma)| < \vartheta$  and hence there is an injection  $\iota \upharpoonright \mathbb{T}_{<\gamma+1} : \mathbb{T}_{<\gamma+1} \longrightarrow \vartheta$  that extends the previous injections and satisfies the above requirement.

**Case 2:**  $\gamma \in S_{<\kappa}^\vartheta$ . Define  $\mathbb{T}(\gamma) = \{t \in {}^\gamma 2 \mid \forall \bar{\gamma} < \gamma \ t \upharpoonright \bar{\gamma} \in \mathbb{T}(\bar{\gamma})\}$ . Then our induction hypothesis implies that for every  $t \in \mathbb{T}_{<\gamma}$  there is a  $u \in \mathbb{T}(\gamma)$  with  $t \subseteq u$ . Moreover, our assumptions imply that  $\gamma^{<\kappa} < \vartheta$  holds and this shows that  $|\mathbb{T}(\gamma)| < \vartheta$ . In particular, we can use our induction hypothesis to find an injection  $\iota \upharpoonright \mathbb{T}_{<\gamma+1} : \mathbb{T}_{<\gamma+1} \longrightarrow \vartheta$  that extends the previous injections and satisfies the above requirements.

Now, assume that  $\gamma \in B$ . In the construction of the function  $a_\gamma : \kappa \longrightarrow \mathbb{T}(\gamma)$ , we distinguish several subclasses:

**Subcase 2.1:**  $\text{Lim}(C_\gamma)$  is unbounded in  $\gamma$ . Set  $a_\gamma(\alpha) = \bigcup \{a_{\bar{\gamma}}(\alpha) \mid \bar{\gamma} \in \text{Lim}(C_\gamma)\}$  for all  $\alpha < \kappa$ . Then our induction hypothesis implies that  $a_\gamma(\alpha) \in \mathbb{T}(\gamma)$  for all  $\alpha < \kappa$ . Given  $\bar{\gamma} \in B \cap \gamma$ , there is  $\gamma_0 \in \text{Lim}(C_\gamma)$  with  $\bar{\gamma} \leq \gamma_0$  and our induction hypothesis shows that there is  $\bar{\kappa} < \kappa$  with  $a_{\bar{\gamma}}(\alpha) \subseteq a_{\gamma_0}(\alpha)$  for all  $\bar{\kappa} \leq \alpha < \kappa$ . This shows that  $a_{\bar{\kappa}}(\alpha) \subseteq a_\gamma(\alpha)$  for all  $\bar{\kappa} \leq \alpha < \kappa$ .

**Subcase 2.2:**  $\gamma \in \text{Lim}(B)$  and  $\text{Lim}(C_\gamma)$  is bounded in  $\gamma$ . Then  $\text{cof}(\gamma) = \omega$  and there is a strictly increasing sequence  $\langle \gamma_n \in B \cap \gamma \mid n < \omega \rangle$  cofinal in  $\gamma$  such that  $\text{Lim}(C_\gamma) \neq \emptyset$  implies  $\gamma_0 = \max(\text{Lim}(C_\gamma))$ . By our induction hypothesis, there is a strictly increasing sequence  $\langle \kappa_n < \kappa \mid n < \omega \rangle$  such that  $\kappa_0 = 0$  and  $a_{\gamma_n}(\alpha) \subseteq a_{\gamma_{n+1}}(\alpha)$  for all  $n < \omega$  and  $\kappa_{n+1} \leq \alpha < \kappa$ . Then  $a_{\gamma_m}(\alpha) \subseteq a_{\gamma_n}(\alpha)$  for all  $m \leq n < \omega$  and  $\kappa_n \leq \alpha < \kappa$ . Given  $n < \omega$  and  $\kappa_n \leq \alpha < \kappa_{n+1}$ , we define  $a_\gamma(\alpha)$  to be some element of  $\mathbb{T}(\gamma)$  that extends  $a_{\gamma_n}(\alpha)$ . In the other case, if  $\sup_{n < \omega} \kappa_n \leq \alpha < \kappa$ , then we define  $a_\gamma(\alpha) = \bigcup \{a_{\gamma_n}(\alpha) \mid n < \omega\} \in \mathbb{T}(\gamma)$ . Then the above choices ensure that  $a_{\gamma_n}(\alpha) \subseteq a_\gamma(\alpha)$  for all  $n < \omega$  and  $\kappa_n \leq \alpha < \kappa$ . In particular, we have  $a_{\gamma_0}(\alpha) \subseteq a_\gamma(\alpha)$  for all  $\alpha < \kappa$  and our induction hypothesis implies that  $a_{\bar{\gamma}}(\alpha) \subseteq a_\gamma(\alpha)$  for all  $\bar{\gamma} \in \text{Lim}(C_\gamma)$  and  $\alpha < \kappa$ . Finally, fix  $\bar{\gamma} \in \bar{B} \cap \gamma$  and pick  $n < \omega$  with  $\bar{\gamma} \leq \gamma_n$ . By our induction hypothesis, there is  $\kappa_n \leq \bar{\kappa} < \kappa$  with  $a_{\bar{\gamma}}(\alpha) \subseteq a_{\gamma_n}(\alpha)$  for all  $\bar{\kappa} \leq \alpha < \kappa$ . By the above computations, we have  $a_{\bar{\gamma}}(\alpha) \subseteq a_\gamma(\alpha)$  for all  $\bar{\kappa} \leq \alpha < \kappa$ .

**Subcase 2.3:**  $\gamma \notin \text{Lim}(B)$  and  $\sup(\gamma \cap \bar{B}) \in \bar{B}$ . Then  $\text{Lim}(C_\gamma)$  is bounded in  $\gamma$  and there are  $\gamma_0 \leq \gamma_1 < \gamma$  such that  $\gamma_0, \gamma_1 \in \bar{B}$ ,  $\gamma_1 = \max(\bar{B} \cap \gamma)$ ,  $\text{Lim}(C_\gamma) \neq \emptyset$  implies  $\gamma_0 = \max(\text{Lim}(C_\gamma))$  and  $\text{Lim}(C_\gamma) = \emptyset$  implies  $\gamma_0 = \gamma_1$ . By our induction hypothesis, there is an  $\kappa_1 < \kappa$  with  $a_{\gamma_0}(\alpha) \subseteq a_{\gamma_1}(\alpha)$  for all  $\kappa_1 \leq \alpha < \kappa$ . Set  $\kappa_0 = 0$  and  $\kappa_2 = \kappa$ . Given  $\alpha < \kappa$ , pick  $i < 2$  with  $\kappa_i \leq \alpha < \kappa_{i+1}$  and define  $a_\gamma(\alpha)$  to be

some element of  $\mathbb{T}(\gamma)$  that extends  $a_{\gamma_i}(\alpha)$ . Given  $\alpha < \kappa$ , we have  $a_{\gamma_0}(\alpha) \subseteq a_\gamma(\alpha)$  and our induction hypothesis implies that  $a_{\tilde{\gamma}}(\alpha) \subseteq a_\gamma(\alpha)$  for all  $\tilde{\gamma} \in \text{Lim}(C_\gamma)$ . Finally, if  $\tilde{\gamma} \in \bar{B} \cap \gamma$ , then  $\tilde{\gamma} \leq \gamma_1$  and our induction hypothesis allows us to find  $\kappa_1 \leq \bar{\kappa} < \kappa$  with  $a_{\tilde{\gamma}} \subseteq a_{\gamma_1}(\alpha)$  for all  $\bar{\kappa} \leq \alpha < \kappa$ . This implies that  $a_{\tilde{\gamma}} \subseteq a_\gamma(\alpha)$  for all  $\bar{\kappa} \leq \alpha < \kappa$ .

**Subcase 2.4:**  $\gamma \notin \text{Lim}(B)$  and  $\tilde{\gamma} = \sup(\gamma \cap \bar{B}) \notin \bar{B}$ . Then  $\text{Lim}(C_\gamma)$  is bounded in  $\gamma$  by  $\tilde{\gamma}$  and  $\lambda = \text{cof}(\tilde{\gamma}) \leq \kappa$ . Pick a strictly increasing continuous sequence  $\langle \gamma_\beta < \tilde{\gamma} \mid \beta < \lambda \rangle$  that is cofinal in  $\tilde{\gamma}$  such that  $\gamma_{\beta+1} \in B$  for all  $\beta < \lambda$ ,  $\text{Lim}(C_\gamma) \neq \emptyset$  implies  $\gamma_0 = \max(\text{Lim}(C_\gamma))$  and  $\text{Lim}(C_\gamma) = \emptyset$  implies  $\gamma_0 = 0$ . By our induction hypothesis, there is a strictly increasing continuous sequence  $\langle \kappa_\beta < \kappa \mid \beta < \lambda \rangle$  such that  $\kappa_0 = 0$  and  $a_{\gamma_{\bar{\beta}}}(\alpha) \subseteq a_{\gamma_{\beta+1}}(\alpha)$  for all  $\kappa_{\beta+1} \leq \alpha < \kappa$  and  $\bar{\beta} \leq \beta < \lambda$  with  $\gamma_{\bar{\beta}} \in B$ . Set  $\kappa_\lambda = \sup_{\beta < \lambda} \kappa_\beta \leq \kappa$ . Fix  $\alpha < \kappa$  and let  $\beta \leq \lambda$  be maximal with  $\kappa_\beta \leq \alpha$ . Then the above definitions and our induction hypothesis imply that

$$\bar{a}_\gamma(\alpha) = \bigcup \{a_{\gamma_{\bar{\beta}}}(\alpha) \mid \bar{\beta} < \max\{\beta, 1\}, \gamma_{\bar{\beta}} \in B\} \in \mathbb{T}_{<\gamma}.$$

Define  $a_\gamma(\alpha)$  to be an element of  $\mathbb{T}(\gamma)$  with  $\bar{a}_\gamma(\alpha) \subseteq a_\gamma(\alpha)$ . Then  $a_{\gamma_0}(\alpha) \subseteq a_\gamma(\alpha)$  for all  $\alpha < \kappa$ . In particular, if  $\tilde{\gamma} \in \text{Lim}(C_\gamma)$ , then our induction hypothesis implies that  $a_{\tilde{\gamma}}(\alpha) \subseteq a_\gamma(\alpha)$  for all  $\alpha < \kappa$ . Finally, pick  $\tilde{\gamma} \in \gamma \cap B$ . Then  $\tilde{\gamma} < \tilde{\gamma}$  and we can find  $\beta < \lambda$  and  $\kappa_{\bar{\beta}+1} \leq \bar{\kappa} < \kappa$  with  $\tilde{\gamma} < \gamma_{\bar{\beta}} \in B$  and  $a_{\tilde{\gamma}}(\alpha) \subseteq a_{\gamma_{\bar{\beta}}}(\alpha)$  for all  $\bar{\kappa} \leq \alpha < \kappa$ . Fix  $\bar{\kappa} \leq \alpha < \kappa$  and let  $\beta \leq \lambda$  be maximal with  $\alpha \geq \kappa_\beta$ . Then  $\beta > \bar{\beta}$  and  $a_{\gamma_{\bar{\beta}}}(\alpha) \subseteq \bar{a}_\gamma(\alpha)$ . This allows us to conclude that  $a_{\tilde{\gamma}}(\alpha) \subseteq a_\gamma(\alpha)$ .

**Case 3:**  $\gamma \in S_\kappa^\vartheta \setminus A$ . Define  $\mathbb{T}(\gamma) = \{t \in {}^\gamma 2 \mid \forall \tilde{\gamma} < \gamma \ t \upharpoonright \tilde{\gamma} \in \mathbb{T}(\tilde{\gamma})\}$ . As above, our assumptions and the induction hypothesis imply that  $|\mathbb{T}(\gamma)| < \vartheta$  and we can find an injection  $\iota \upharpoonright \mathbb{T}_{<\gamma+1} : \mathbb{T}_{<\gamma+1} \rightarrow \vartheta$  with the desired properties.

Now, assume that  $\gamma \in B$ . Since  $\text{cof}(\gamma) > \omega$ , we know that  $\text{Lim}(C_\gamma) \subseteq B$  is unbounded in  $\gamma$  and our induction hypothesis shows that

$$a_\gamma(\alpha) = \bigcup \{a_{\tilde{\gamma}}(\alpha) \mid \tilde{\gamma} \in \text{Lim}(C_\gamma)\} \in \mathbb{T}(\gamma)$$

for all  $\alpha < \kappa$ . Moreover, Our induction hypothesis directly implies that this sequence has the desired properties.

**Case 4:**  $\gamma \in A$ . Since  $\text{cof}(\gamma) > \omega$ , there is a strictly increasing continuous sequence  $\langle \gamma_\alpha \in \text{Lim}(C_\gamma) \mid \alpha < \kappa \rangle$  cofinal in  $\gamma$ . Pick a maximal antichain  $\mathcal{A}$  in  $\mathbb{T}_{<\gamma}$  such that  $\mathcal{A}$  is equal to the  $(\iota \upharpoonright \mathbb{T}_{<\gamma})$ -preimage of  $A_\gamma$  if this preimage is a maximal antichain in  $\mathbb{T}_{<\gamma}$ . Let  $\mathbb{T}_{\mathcal{A}}$  denote the set of all  $t \in \mathbb{T}_{<\gamma}$  with  $s \leq_{\mathbb{T}} t$  for some  $s \in \mathcal{A}$ . Given  $t \in \mathbb{T}_{\mathcal{A}}$ , our induction hypothesis allows us to find  $u_t \in {}^\gamma 2$  such that  $t \subseteq u_t$  and  $u_t \upharpoonright \tilde{\gamma} \in \mathbb{T}(\tilde{\gamma})$  for all  $\tilde{\gamma} < \gamma$ . We define  $\mathbb{T}(\gamma) = \{u_t \mid t \in \mathbb{T}_{\mathcal{A}}\}$ . Then our induction hypothesis implies that  $|\mathbb{T}(\gamma)| < \vartheta$  and we can find a suitable injection  $\iota \upharpoonright \mathbb{T}_{<\gamma+1} : \mathbb{T}_{<\gamma+1} \rightarrow \vartheta$ . Moreover, if the  $(\iota \upharpoonright \mathbb{T}_{<\gamma})$ -preimage of  $A_\gamma$  is a maximal antichain in  $\mathbb{T}_{<\gamma}$ , then the above construction ensures that for every  $u \in \mathbb{T}(\gamma)$ , there is a  $t \in \mathbb{T}_{<\gamma}$  with  $t \subseteq u$  and  $\iota(t) \in A_\gamma$ . Finally, the maximality of  $\mathcal{A}$  in  $\mathbb{T}_{<\gamma}$  implies that for every  $\alpha < \kappa$ , there is an  $\bar{a}_\gamma(\alpha) \in \mathbb{T}_{\mathcal{A}}$  with  $a_{\gamma_\alpha}(\alpha) \subseteq \bar{a}_\gamma(\alpha)$ . Define  $a_\gamma(\alpha) = u_{\bar{a}_\gamma(\alpha)} \in \mathbb{T}(\gamma)$  for all  $\alpha < \kappa$ . Pick  $\tilde{\gamma} \in \bar{B} \cap \gamma$ . Then we can find  $\bar{\alpha} < \bar{\kappa} < \kappa$  such that  $\tilde{\gamma} < \gamma_{\bar{\alpha}}$  and  $a_{\tilde{\gamma}}(\alpha) \subseteq a_{\gamma_{\bar{\alpha}}}(\alpha)$  for all  $\bar{\kappa} \leq \alpha < \kappa$ . Fix  $\bar{\kappa} \leq \alpha < \kappa$ . Then  $\bar{\alpha} < \alpha$ ,  $\gamma_{\bar{\alpha}} \in B \setminus A$  and  $\gamma_{\bar{\alpha}} \in \text{Lim}(C_{\gamma_\alpha})$ . Using our induction hypothesis, we can conclude that  $a_{\tilde{\gamma}}(\alpha) \subseteq a_{\gamma_{\bar{\alpha}}}(\alpha) \subseteq a_{\gamma_\alpha}(\alpha) \subseteq \bar{a}_\gamma(\alpha) \subseteq a_\gamma(\alpha)$ .

**Case 5:**  $\gamma \in S_{\kappa^+}^\vartheta$ . Then  $A \cap \text{Lim}(C_\gamma) = \emptyset$  and  $\text{otp}(C_\gamma) = \kappa^+$ . Given  $t \in \mathbb{T}_{<\gamma}$ , this shows that we can use our induction hypothesis to construct  $u_t \in {}^\gamma 2$  with  $t \subseteq u_t$

and  $u_t \upharpoonright \bar{\gamma} \in \mathbb{T}(\bar{\gamma})$  for all  $\bar{\gamma} < \gamma$ . Next, define  $a_\gamma(\alpha) = \bigcup \{a_{\bar{\gamma}}(\alpha) \mid \bar{\gamma} \in \text{Lim}(C_\gamma)\}$  for each  $\alpha < \kappa$ . Given  $\alpha < \kappa$ , our induction hypothesis shows that  $a_\gamma(\alpha) \in {}^\gamma 2$  and  $a_\gamma(\alpha) \upharpoonright \bar{\gamma} \in \mathbb{T}(\bar{\gamma})$  for all  $\bar{\gamma} < \gamma$ . We define  $\mathbb{T}(\gamma) = \{u_t \mid t \in \mathbb{T}_{<\gamma}\} \cup \{a_\gamma(\alpha) \mid \alpha < \kappa\}$ . Then our induction hypothesis implies that  $|\mathbb{T}(\gamma)| < \vartheta$  and we can find a suitable injection  $\iota \upharpoonright \mathbb{T}_{<\gamma+1} : \mathbb{T}_{<\gamma+1} \rightarrow \vartheta$ . Finally, our induction hypothesis implies that the function  $a_\gamma : \kappa \rightarrow \mathbb{T}(\gamma)$  has the desired properties.

This completes the inductive construction of  $\mathbb{T}$  and  $\iota : \mathbb{T} \rightarrow \vartheta$ .

**Claim.**  $\mathbb{T}$  is a  $\vartheta$ -Souslin tree.

*Proof of the Claim.* By the above construction,  $\mathbb{T}$  is a tree of height  $\vartheta$ . Let  $\mathcal{A}$  be a maximal antichain in  $\mathbb{T}$ . Using the properties of  $\iota$ , we can construct a club  $C$  in  $\vartheta$  with the property that  $\text{ran}(\iota \upharpoonright \mathbb{T}_{<\gamma}) = \gamma \cap \text{ran}(\iota)$  and  $\mathcal{A} \cap \mathbb{T}_{<\gamma}$  is a maximal antichain in  $\mathbb{T}_{<\gamma}$  for all  $\gamma \in C$ . Then there is  $\gamma \in A \cap C$  such that  $A_\gamma$  is equal to the  $\iota$ -image of  $\mathcal{A} \cap \mathbb{T}_{<\gamma}$ . Then the  $(\iota \upharpoonright \mathbb{T}_{<\gamma})$ -preimage of  $A_\gamma$  is a maximal antichain in  $\mathbb{T}_{<\gamma}$  and for every  $u \in \mathbb{T}(\gamma)$ , there is a  $t \in \mathbb{T}_{<\gamma}$  with  $t \subseteq u$  and  $\iota(t) \in A_\gamma$ . This shows that  $\mathcal{A} \subseteq \mathbb{T}_{<\gamma}$  and  $|\mathcal{A}| < \vartheta$ .  $\square$

Given  $\gamma < \vartheta$  and  $\alpha < \kappa$ , set  $\delta = \min(\bar{B} \setminus \gamma)$  and let  $b_\gamma(\alpha)$  denote the unique element of  $\mathbb{T}(\gamma)$  with  $b_\gamma(\alpha) \subseteq a_\delta(\alpha)$ . By the above constructions, if  $\bar{\gamma} < \gamma < \vartheta$ , then there  $\bar{\kappa} < \kappa$  such that  $b_{\bar{\gamma}}(\alpha) \subseteq b_\gamma(\alpha)$  for all  $\bar{\kappa} \leq \alpha < \kappa$ . This shows that the resulting sequence  $\langle b_\gamma(\alpha) : \kappa \rightarrow \mathbb{T}(\gamma) \mid \gamma < \vartheta \rangle$  is a  $\kappa$ -ascent path through  $\mathbb{T}$ .  $\square$

In the remainder of this section, we will combine the above results with a theorem of Larson from [13] on the preservation of PFA under  $<\omega_2$ -directed closed forcings to derive the statements of Theorem 1.14.

*Proof of Theorem 1.14.* Assume that PFA holds.

(i) Assume that  $\theta > \omega_1$  and  $\mathbb{T}$  contains an ascending path of width  $\omega$ . Then Theorem 5.3 implies that  $\text{ISP}(\omega_2)$  holds and Lemma 5.4 shows that every narrow  $\theta$ -system of width  $\omega$  has a cofinal branch. This shows that  $\mathbb{T}$  has a cofinal branch.

(ii) Let  $\dot{\mathbb{P}}$  be a  $\text{Col}(\omega_3, 2^{\omega_2})$ -name for the partial order given by Lemma 5.6 for  $\omega_2$  and let  $G * H$  be  $\text{Col}(\omega_3, 2^{\omega_2}) * \dot{\mathbb{P}}$ -generic over  $V$ . Since the partial order  $\text{Col}(\omega_3, 2^{\omega_2}) * \dot{\mathbb{P}}$  is  $<\omega_2$ -directed closed, the results of [13] show that PFA holds  $V[G, H]$  and this implies that  $2^{\omega_1} = \omega_2$  holds in  $V[G, H]$ . Moreover, since  $\dot{\mathbb{P}}^G$  is  $(\omega_2 + 1)$ -strategically closed in  $V[G]$ , we have

$$(2^{\omega_2})^{V[G, H]} = (2^{\omega_2})^{V[G]} = \omega_3^{V[G]} = \omega_3^{V[G, H]}.$$

Finally, there is a  $\square_{\omega_2}^B$ -sequence in  $V[G, H]$  and Theorem 5.8 shows that there is an  $\omega_3$ -Souslin tree with an  $\omega_1$ -ascent path in  $V[G, H]$ .

(iii) Assume that  $\kappa$  is strongly compact and let  $G$  be  $\text{Col}(\omega_2, <\kappa)$ -generic over  $V$ . Then the results of [13] show that PFA holds in  $V[G]$ . In  $V[G]$ , if  $\mathbb{T}$  is a tree of regular height greater than  $\omega_2$  containing an ascending path of width  $\omega_1$ , then Theorem 1.13.(v) implies that  $\mathbb{T}$  has a cofinal branch.  $\square$

## 6. OPEN QUESTIONS

We close this paper with a list of questions raised by the above results. The first question is motivated by the assumptions of Corollary 1.7 and asks whether these assumption are necessary for successors of singular cardinals.

**Question 6.1.** *Given a singular cardinal  $\nu$  and a cardinal  $\text{cof}(\nu) \leq \lambda < \nu$ , is it possible that a special tree of height  $\nu^+$  contains an ascending path of width  $\lambda$ ?*

Given an uncountable cardinal  $\kappa$  with  $\kappa = \kappa^{<\kappa}$ , Theorem 1.9 shows that a tree of height  $\kappa^+$  is specializable if and only if it contains no ascending paths of width less than  $\kappa$ . It is natural to ask whether this equivalence holds without the cardinal arithmetic assumption.

**Question 6.2.** *If  $\kappa$  is an uncountable regular cardinal and  $\mathbb{T}$  is a tree of height  $\kappa^+$  that does not contain an ascending path of width less than  $\kappa$ , is  $\mathbb{T}$  specializable?*

The following special case of the above question is motivated by Theorem 1.14.

**Question 6.3.** *Assume that PFA holds. Is every tree of height  $\omega_2$  without a cofinal branch specializable?*

A negative answer to Question 6.2 would leave open the possibility that specializable trees can be characterized by some combinatorial property.

**Question 6.4.** *If  $\kappa$  is an uncountable regular cardinal, is the class of specializable trees of height  $\kappa^+$  definable in  $V$ ?*

The proof of Theorem 1.11 in Section 2 shows that, under the cardinal arithmetic assumptions of the theorem, the existence of an ascending path through  $\mathbb{T}$  is equivalent to the existence of such a path with seemingly stronger compatibility properties. It is not known to the author whether this equivalence holds without the assumptions of the theorem.

**Question 6.5.** *If the tree  $\mathbb{T}$  contains an ascending sequence of width  $\lambda$ , is there a sequence  $\langle b_\gamma : \lambda \rightarrow \mathbb{T}(\gamma) \mid \gamma < \theta \rangle$  such that for all  $\bar{\gamma} < \gamma < \theta$ , there is an  $\alpha < \lambda$  with  $b_{\bar{\gamma}}(\alpha) <_{\mathbb{T}} b_\gamma(\alpha)$ ?*

Theorem 1.13,(iii) shows that it is possible to obtain a model in which every tree of height  $\omega_2$  that contains an ascending path of width  $\omega$  has a cofinal branch. The discussion following the proof of Corollary 4.13 in Section 4 shows that this statement implies that  $\omega_2$  is Mahlo in  $L$ . Therefore it is natural to ask for the exact consistency strength of this statement.

**Question 6.6.** *Does the assumption that every tree of height  $\omega_2$  that contains an ascending path of width  $\omega$  has a cofinal branch imply that  $\omega_2$  is a weakly compact cardinal in  $L$ ?*

The obvious strategy answer Question 6.6 in the positive is to show that the extra assumptions on the  $\square(\theta)$ -sequence in Theorem 4.12 are not necessary.

**Question 6.7.** *Does the existence of a  $\square(\theta)$ -sequence imply the existence of an  $\theta$ -Aronszajn tree that contains an ascending path of width  $\lambda$  with  $\lambda^+ < \theta$ ?*

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