The Quandry of Quandles: The Borel Completeness of a Knot Invariant

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Abstract

The isomorphism type of the knot quandle introduced by Joyce is a complete invariant of tame knots. Whether two quandles are isomorphic is in practice difficult to determine; we show that this question is provably hard: isomorphism of quandles is Borel complete. The class of tame knots, however, is trivial from the perspective of Borel reducibility, suggesting that equivalence of tame knots may be reducible to a more tractable isomorphism problem.

1 Introduction

Left distributivity arises in the study of many well-known mathematical objects such as groups, knots and braids, and also in the study of large cardinal embeddings in set theory. Specifically, left distributive algebras are structures with one binary operation $*$ satisfying the left self-distributivity law $a * (b * c) = (a * b) * (a * c)$. Familiar examples include the conjugation operation on any group and the implication operation on any Boolean algebra; symmetric spaces in differential geometry provide further examples [1]. The first nontrivial example of a free left distributive algebra on one generator is due to Laver [15], who showed that the algebra generated by a certain elementary embedding under the application operation is such an algebra (the existence of these embeddings is one of the strongest known set-theoretic axioms).

Other interesting classes of structures are obtained by adding further algebraic axioms to the left distributive law, with an important case being the quandles. Quandles are left distributive algebras satisfying $a*a = a$ and such that for every a and c in the algebra there is a unique b such that $a * b = c$. It is quandles that are the focus of this note. Isomorphism type of quandles is a complete invariant of knots, and we prove that isomorphism of quandles is, from the perspective of Borel reducibility, fundamentally difficult (Borel complete). After first offering an introduction

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to quandles and Borel reducibility, we present the technical preliminaries in Section 2, give the main result and corollaries in the next section, and discussion in the final section.

In his doctoral thesis, and published in [12], Joyce rediscovered quandles and coined the term *quandle*. There he established many foundational relationships, including those between quandles and group conjugation and quandles and knots. Indeed he showed that the equational theory of quandles is precisely the equational theory of the conjugation operation: any identity true in every group with its conjugation operation is also true in every quandle, and hence provable from the quandle axioms. The three quandle axioms may also be viewed as algebraic versions of the familiar Reidemeister moves for translating between different regular projections of equivalent knots. One may consequently associate to any tame knot a quandle generated by the arcs of the knot and with identities dictated by the crossings; notice that all such quandles are finitely presented. In addition to defining the knot quandle in this way, Joyce showed that these quandles in fact constitute complete invariants for tame knots: two tame knots are equivalent if and only if their associated quandles are isomorphic. (Tame knots essentially correspond to one's intuitive notion of finite knots in three-dimensional space, and in particular are not assumed to be endowed with an orientation.)

The complexity of classification problems and the study of complete invariants for structures have emerged as major themes in set theory. Broadly, a classification can be thought to assign mathematical objects of one type — considered up to isomorphism or some other such equivalence relation — to mathematical objects of another type (again up to an equivalence relation), where the former act as invariants. Frequently the objects in question, both those to be classified and the invariants, can be encoded by real numbers. For example, countable structures with underlying set $\mathbb{N}$, such as groups, rings, and indeed left distributive algebras, can be encoded in a natural way by sets of finite tuples of natural numbers, and hence by reals. Classification then amounts to finding a reasonably definable map from the reals encoding the structures to the reals encoding the invariants that respects the relevant equivalence relations. Of course, the "reasonably definable" is important here — a non-constructive proof of the existence of such a map using, say, the Axiom of Choice should not be considered a classification. A natural way to exclude such uninformative maps would be to require the map to be continuous, but this interpretation is too restrictive to be practical. The more liberal constraint that the map be Borel, however, permits almost all constructions that arise in practice whilst being restrictive enough to obtain meaningful theorems about the framework.

Classifying structures using Borel maps between sets of encoding reals gives rise to the notion of Borel reducibility. Given two equivalence relations E and F on real numbers, say that E is Borel reducible to F, written $E \leq_B F$, if there is a Borel function f from $\mathbb{R}$ to $\mathbb{R}$ such that for all x and y in $\mathbb{R}$, $x \mathrel{E} y$ holds if and only if $f(x) \mathrel{F} f(y)$ holds. Establishing that one equivalence relation is *not* Borel reducible to another has been used in a number of cases to show that a classification problem is impossible to resolve. For example, Farah, Toms, and Törnquist [6] used this analysis to show that unital simple separable nuclear C^*-algebras are not classifiable by countable structures (note that each adjective makes the theorem stronger), and Foreman, Rudolph, and Weiss [8] showed that ergodic measure-preserving transformations of the unit interval are not classifiable by countable structures (and indeed much more). For more on this area see, for example, Hjorth's book [11]. Within the scope of knot theory, Kulikov [14] has recently shown that the class of all knots — including, for example, wild knots with infinitely many crossings — is not classifiable by countable structures.

Against this background it is natural to ask: what is the Borel complexity of the isomorphism relation on the most general class of countable left distributive structures, the countable left distributive algebras? This question was indeed posed to the second author by Matt Foreman. In this note we show that it has the maximum possible complexity for an isomorphism relation on countable structures: in the standard terminology introduced in the seminal paper of Friedman and Stanley [9], isomorphism of left distributive algebras is *Borel complete*. Moreover the same is true for the subclasses of racks, quandles, and keis (see Section 2 for definitions). We show directly that isomorphism of keis (Definition 1.4) is Borel complete; the result for the other, more general classes follows. We also show that the related class of expansions of left distributive algebras satisfying the set of axioms Laver [16] denoted by Σ (Definition 2) is Borel complete, although the argument proceeds differently.

Knot theorists express some dissatisfaction with quandles as knot invariants because of the difficulty in determining whether two quandles are isomorphic. This difficulty is perhaps not surprising: our result says that isomorphism of arbitrary countable quandles is Borel complete. By contrast, tame knots can reasonably be encoded up to equivalence by equivalence classes of natural numbers rather than reals, and hence are trivial in the context of Borel reducibility. It is therefore reasonable to hope that a complete invariant for knots that is simpler than the quandle (in terms of Borel reducibility) might be discovered. Of course, the subclass of those quandles arising from tame knots is countable up to quandle isomorphism. Furthermore, as previously remarked, all quandles from tame knots are finitely presented; the class of finitely presented quandles also has only countably many members up to isomorphism, and so is trivial in Borel reducibility terms. Finitely presented quandles are thus optimal in this sense as invariants for tame knots, but their finite presentability is crucial to this fact. We speculate that a non-Borel complete class of structures with a definition that does not depend on the cardinality of the presentation of the structure may provide complete invariants for tame knots which are in practice easier to test for isomorphism.

2 Preliminaries

As we will be discussing the related classes of left distributive algebras, racks, quandles, and keis, we begin by giving some intuition for them. These classes of structures can usefully be understood in terms of the behaviour of the action of left multiplication by an element of the algebra. For structures with underlying set A and binary operation $*$, and for each a in A, denote by m_a the map from A to A that acts by multiplication on the left by a, that is, $m_a(b) = a * b$. Then *left distributive algebras* are those for which m_a is a homomorphism from A to itself for each a in A. A *rack* is a left distributive algebra in which each m_a is an automorphism (indeed Brieskorn [2] referred to racks as *automorphic sets*). In a *quandle*, m_a an automorphism and

a is a fixed point of m_a for each a in A. Finally, a *kei* (also called an *involutory quandle*) is a quandle such that each m_a is its own inverse. The word *quandle* was introduced in 1982 by Joyce [12], and *kei* in 1943 by Takasaki [21], who introduced several variants of keis, many of them reflecting symmetries of geometric configurations of points in the plane. While together at Cambridge, Wraith and Conway investigated what remains of a group when all the other structure is neglected and only conjugation remains; as a pun on Wraith's name and these wrecked groups, Conway called them *wracks* [22]. Fenn and Rourke [7] took this term, adjusted the spelling to rack, and gave it the present precise meaning.

Formally, these structures can be defined using the following axioms:

i. For every a, b, and c in A, $a * (b * c) = (a * b) * (a * c)$.

ii. For all a and c in A, there is a unique b in A such that $a * b = c$.

iii. For every a in A, $a * a = a$.

iv. For all a and b in A, $a * (a * b) = b$.

Definition 1. For a set A with one binary operation $*$ (an *algebra*), define:

1. A *left distributive algebra* is an algebra satisfying axiom (i).
2. A *rack* is an algebra satisfying axioms (i) and (ii).
3. A *quandle* is an algebra satisfying axioms (i), (ii) and (iii).
4. A *kei* is an algebra satisfying axioms (i), (ii), (iii) and (iv).

There are a number of choices to be made in presenting the above definitions. Instead of using axiom (ii), one can formulate racks using a second operation $\bar{*}$ such that the function $m_a : b \mapsto a * b$ is inverse to the function $b \mapsto a \,\bar{*}\, b$: formallly, one requires that for all a and b, $a \,\bar{*}\, (a * b) = a * (a \,\bar{*}\, b) = b$ holds. This has the advantage of eliminating the existential quantifier. Whether to consider self distributive structures as left distributive, like we do here, or right distributive (with axioms (ii) and (iv) reformulated for right multiplication) is an arbitrary choice. Many relevant references on racks, quandles, and keis use right distributivity; we chose left distributivity in order to easily view these classes of structures as subclasses of the left distributive algebras.

There is another well-studied left distributive structure, this one with two operations: the left distributive operation $*$ and another operation $\circ$ that behaves like composition. These algebras were first studied by Laver [15] as algebras of large cardinal embeddings in which the operation $\circ$ is in fact composition.

Definition 2. We denote by Σ the following collection of four identities.

$$\begin{aligned} a \circ (b \circ c) &= (a \circ b) \circ c \\ (a \circ b) * c &= a * (b * c) \\ a * (b \circ c) &= (a * b) \circ (a * c) \\ (a * b) \circ a &= a \circ b \end{aligned}$$

Note that left distributivity follows from the second and fourth identities via the equalities $a * (b * c) = (a \circ b) * c = ((a * b) \circ a) * c = (a * b) * (a * c)$. Dehornoy refers to algebras satisfying

Σ as *LD-monoids*; we use Laver's original phrase "algebras satisfying Σ" to avoid any potential confusion with other uses of "monoid."

If $\circ$ is a group operation on A then the fourth equational condition of Σ determines that the other operation $*$ must be the conjugation operation $a * b = a \circ b \circ a^{-1}$. Taking $*$ to denote conjugation in the group in question, it is straightforward to check that the other identities of Σ are also satisfied, so any group with its multiplication and conjugation operations is an algebra satisfying Σ.

Laver showed, among other things, that Σ is a conservative extension of the left distributive law [15]. Thus any free left distributive algebra may be expanded to a free algebra on the same generators satisfying Σ: any identity on elements of the free left distributive algebra will hold in the algebra satisfying Σ if and only if it is a consequence of the left distributive law. For more on this, the linearity of several orderings on the free left distributive algebra (from the large cardinal hypothesis), and a normal form for terms in the free left distributive algebra, see [15] and [16]. For a simpler proof and fuller account of the theory of left distributive algebras, see [17]. Using braid groups Dehornoy showed within the standard axioms of set theory that the above-mentioned orderings on the free left distributive algebra are linear [4]; Dehornoy has also contributed substantially to the literature on algebras satisfying Σ. See, for example, [5].

We now move on to preliminaries regarding Borel reducibility. Recall that a subset of a topological space is *Borel* if it lies in the least σ-algebra containing the open sets, and that a function between two topological spaces is Borel if the inverse image of any Borel set (or equivalently, of any open set) is Borel. Thus, to discuss Borel reducibility between classes of countable structures, we first define a topology on each of these classes. We briefly sketch this definition here, and refer the reader to Section 2.3 of Hjorth's book [11] for further details.

We exclusively consider countable structures, and so may assume that each structure has underlying set $\mathbb{N}$. Furthermore all of the classes of structures we consider are first-order, namely, the structures have finitely many relations and operations, and the class is defined by formulas involving these relations and operations. The relations and operations of a structure in one of these classes can thus be represented by a set of tuples from $\mathbb{N}$. Indeed we follow the common practice of identifying a directed graph $(\mathbb{N}, E)$ (with vertex set $\mathbb{N}$) with the set $\{(m, n) \mid m\ E\ n\} \subseteq \mathbb{N}^2$, and we may identify an algebra $(\mathbb{N}, *)$ with the set $\{(\ell, m, n) \mid \ell * m = n\} \subset \mathbb{N}^3$. The space of countable structures for a given signature with finitely many operation and relation symbols can thus be identified with a subset of Cantor space via the usual identification of a power set $\mathcal{P}(X)$ with the space of characteristic functions 2^X; the set X here is a product of sets of the form $\mathbb{N}^k$, one for each relation and operation, and is in particular countable. The topology considered on these classes is the standard topology on the Cantor space. Note that a clopen subbase for this topology is given by the sets defined by determining a single "bit" from 2^X — for example, on the space of countable algebras with underlying set $\mathbb{N}$, the subbase is the collection as ℓ, m, and n vary over $\mathbb{N}$ of all sets either of the form $\{(\mathbb{N}, *) \mid \ell * m = n\}$ or of the form $\{(\mathbb{N}, *) \mid \ell * m \neq n\}$.

We deviate from this conventional framework in one detail: for expositional clarity, the keis that we construct will have underlying set $\mathbb{N} \times \{0, 1\}$ rather than $\mathbb{N}$. However, this discrepancy can be easily overcome using the canonical identification of $\mathbb{N} \times \{0, 1\}$ with $\mathbb{N}$ via the map $(n, i) \mapsto 2n + i$.

Note that the Cantor space 2^X with X countable is a separable topological space (that is, it has a countable dense set) and may be endowed with a complete metric: identifying X with $\mathbb{N}$, let $d(x, y) = 2^{-n}$ where n is least such that $x(n) \neq y(n)$. Separable, completely metrizable spaces such as 2^X and $\mathbb{R}$ are known as *Polish spaces*. As outlined in the Introduction, we have the following standard definitions.

Definition 3. Let X and Y be Polish spaces, E an equivalence relation on X, and F an equivalence relation on Y. We say that E is *Borel reducible to* F, written $E \leq_B F$, if there is a Borel function f from X to Y such that for all x and x' in X, $x \mathrel{E} x'$ holds (that is, x is E-equivalent to x') if and only if $f(x) \mathrel{F} f(x')$ holds.

We say that E is *continuously reducible* to F, written $E \leq_c F$, if there is a continuous function f from X to Y such that for all x and x' in X, $x \mathrel{E} x'$ if and ony if $f(x) \mathrel{F} f(x')$.

If F is the isomorphism relation for a first-order class of countable structures for a finite signature each with underlying set $\mathbb{N}$, we say F is *Borel complete* if every other such class has isomorphism relation Borel reducible to F.

Continuous maps are of course Borel, and all maps we construct in the sequel will be continuous so in particular Borel.

3 Keis are Borel Complete

It is folklore that the class of countable irreflexive directed graphs is Borel complete — see Section 13.1 of Gao's book [10] for a proof of the stronger statement that the subclass of countable irreflexive symmetric graphs is Borel complete. The general strategy of this section is to construct a kei from an arbitrary irreflexive directed graph, and then to show that the resulting keis are isomorphic if and only if the original graphs are isomorphic. Since the map taking each irreflexive directed graph to the corresponding kei will be Borel (indeed, continuous), this will establish that the class of countable keis is also Borel complete. To this end we shall describe how to build what Kamada [13] calls a *dynamical quandle*; the specific dynamical quandles we construct will in fact be keis.

In all of the sequel we exclusively discuss graphs that are irreflexive and directed, but for the sake of the casual reader, we will repeat these hypotheses each time they are used.

Let A be a set and τ a bijection from A to itself. Let φ be a map from A to the power set $\mathcal{P}(A)$ such that for every $a \in A$, $\varphi(a)$ contains a, $\varphi(a)$ is closed under τ and τ^{-1}, and $\varphi(a) = \varphi(\tau a)$. We will refer to such maps φ as τ-*replete*. Kamada observes [13, Theorem 4] that with the operation $*$ defined by

$$a * b = \begin{cases} b & \text{if } a \in \varphi(b) \\ \tau b & \text{if } a \notin \varphi(b), \end{cases}$$

the structure $(A,*)$ is a quandle. Kamada uses an equivalent definition with a function θ defined on τ-orbits rather than our orbit-invariant function φ on elements of A. Axioms (ii) and (iii) of Definition 1 are immediate from the assumptions on φ, and (i) follows by checking cases:

$$a*(b*c) = (a*b)*(a*c) = \begin{cases} c & \text{if } a \in \varphi(c) \text{ and } b \in \varphi(c) \\ \tau c & \text{if } a \in \varphi(c) \text{ and } b \notin \varphi(c) \\ \tau c & \text{if } a \notin \varphi(c) \text{ and } b \in \varphi(c) \\ \tau^2 c & \text{if } a \notin \varphi(c) \text{ and } b \notin \varphi(c). \end{cases}$$

Moreover, if τ is an involution, then clearly axiom (iv) also holds and so the quandle is a kei. Following Kamada, but using our φ rather than Kamada's θ, we call this $(A,*)$ the *quandle derived from* (A,τ) *relative to* φ. Kamada named the objects so constructed *dynamical quandles*, in line with a view of the pair (A,τ) as a dynamical system, and we shall call those dynamical quandles that are keis *dynamical keis*.

To encode an irreflexive directed graph $G=(V,E)$ into a kei Q_G, we use the dynamical quandle construction with underlying set a *pair* of copies of the vertex set V of G. Our involution τ simply switches between the two copies of the vertex set, and the function φ corresponds to choosing the set of neighbours (in one direction) for each vertex of G, irrespective of which copy of V the vertices lie in.

Definition 4. Suppose $G=(V,E)$ is an irreflexive directed graph. Let τ be the involution on $V\times\{0,1\}$ taking $(v,0)$ to $(v,1)$ and $(v,1)$ to $(v,0)$ for every v in V. Let $\bar{\varphi}_G$ be the function from V to $\mathcal{P}(V)$ defined by $u \in \bar{\varphi}_G(v)$ if and only if $u\,E\,v$ or $u=v$. Let φ_G from $V\times\{0,1\}$ to $\mathcal{P}(V\times\{0,1\})$ be the function obtained from $\bar{\varphi}_G$ by ignoring second coordinates: $(u,i)\in\varphi_G(v,j)$ if and only if $u\in\bar{\varphi}_G(v)$, that is, if and only if $u\,E\,v$ or $u=v$. Note that φ_G is τ-replete. The *kei* Q_G *associated to* G is the quandle derived from $(V\times\{0,1\},\tau)$ relative to φ_G, and we denote the operation on Q_G by $*_G$.

Thus, Q_G is a kei on underlying set $V\times\{0,1\}$ with operation $*$ such that $(u,i)*(v,j)$ equals (v,j) if there is an edge from u to v in G or if $u=v$, and $(u,i)*(v,j)$ is $(v,1-j)$ otherwise.

We now begin toward Theorem 8, which says that the dynamical keis Q_G and $Q_{G'}$ constructed from graphs G and G' are isomorphic if and only if the graphs G and G' are isomorphic. First we prove the existence of a particular, useful involution of the kei Q_G (Lemma 5).

Lemma 5. *For every irreflexive directed graph G with underlying set V and every $W\subseteq V$, the function $I_W : Q_G \to Q_G$ defined by*

$$I_W(v,j) = \begin{cases} (v,j) & \text{if } v \in W \\ (v,1-j) & \text{if } v \notin W \end{cases}$$

is an involution of Q_G.

Proof. By inspection I_W is a bijection and moreover $(I_W)^2$ is the identity map. To see that I_W respects the quandle operation $*$ of Q_G, we must verify that $I_W((u,i)*(v,j)) = I_W(u,i)*$

$I_W(v,j)$. Note that for each $(v,j) \in Q_G$, either both of $(u,0)$ and $(u,1)$ are in $\varphi_G(v,j)$ or neither is, so

$$(u,i) * (v,j) = (I_W(u,i)) * (v,j) = \begin{cases} (v,j) & \text{if } (u,i) \in \varphi(v,j) \\ (v,1-j) & \text{if } (u,i) \notin \varphi(v,j). \end{cases}$$

So

$$I_W((u,i) * (v,j)) = \begin{cases} I_W(v,j) & \text{if } (u,i) \in \varphi(v,j) \\ I_W(v,1-j) & \text{if } (u,i) \notin \varphi(v,j) \end{cases}$$

and

$$I_W((u,i)) * I_W(v,j) = \begin{cases} I_W(v,j) & \text{if } (u,i) \in \varphi(I_W(v,j)) = \varphi((v,j)) \\ (v,1-j) = I_W(v,1-j) & \text{if } v \in W \text{ and } (u,i) \notin \varphi((v,j)) \\ (v,j) = I_W(v,1-j) & \text{if } v \notin W \text{ and } (u,i) \notin \varphi((v,j)). \end{cases}$$

Thus it is established that I_W is a homomorphism, indeed an involution of Q_G. □

A slicker if less direct proof of Lemma 5 is to consider the graph G' on $V \dot\cup \{v_0\}$ (where $\dot\cup$ denotes disjoint union) with $G' \restriction V = G$ and $v_0 \; E \; v$ if and only if v is in W for each v in V. Then $Q_{G'} \restriction V \times \{0,1\} = Q_G$, and $m_{v_0} \restriction Q_G = I_W$.

The keis constructed in Definition 4 are in fact quite general dynamical keis. Indeed the only extra constraint we need on dynamical keis to get a kei Q_G associated to a graph G is that the involution τ has no fixed points.

Definition 6. A kei $(A,*)$ is called a *folded kei*[1] if there is an involution τ of A with no fixed points and a τ-replete function φ such that $(A,*)$ is the quandle derived from (A,τ) relative to φ.

By definition the kei Q_G associated to any graph G is a folded kei. As alluded to above, we also have a converse to this.

Proposition 7. *Every folded kei is isomorphic to a kei of the form Q_G for some irreflexive directed graph G.*

Proof. Let $(A,*)$ be a folded kei, and in particular suppose $(A,*)$ is the quandle derived from (A,τ) relative to φ for τ an involution of A without fixed points and φ a τ-replete function from A to $\mathcal{P}(A)$. Choose a subset V of A such that for each pair $\{a, \tau a\}$ of elements of A, exactly one of a and τa is in V, and express A as the disjoint union $A = V \cup \{\tau v \mid v \in V\}$. For each v in V, let $\bar\varphi(v)$ denote the set $\varphi(v) \cap V$; since $(A,*)$ is the quandle derived from (A,τ) relative to φ we have that $\bar\varphi(v)$ is the set of u in V such that $u * v = v$ (this $\bar\varphi$ will be $\bar\varphi_G$ as in Definition 4 for the graph G we now construct). Take the directed graph G on vertex set V with edge relation defined by $u \; E \; v$ if and only if $u \in \bar\varphi(v)$ holds. Then it is straightforward to check that the map from Q_G to A taking $(v,0)$ to v and $(v,1)$ to τv is an isomorphism of keis. □

We will now state the main result.

[1] In baking, one *folds* ingredients to achieve complete mixing with minimal disruption.

Theorem 8. *For irreflexive directed graphs G and G' and the associated keis Q_G and Q'_G, $G \cong G'$ if and only if $Q_G \cong Q_{G'}$*

Proof. One direction is a fairly straightforward observation:

Remark. Isomorphic irreflexive directed graphs have isomorphic associated keis.

Proof of Remark. Recall that a graph isomorphism is a bijection between vertices that preserves both the edge relation and the failure of the edge relation. Given graphs $G = (V, E)$ and $G' = (V', E')$ with an isomorphism $h : G \to G'$ between them, $u\ E\ v$ in G if and only if $h(u)\ E'\ h(v)$ in G', so u is in $\bar{\varphi}_G(v)$ if and only if $h(u)$ is in $\bar{\varphi}_{G'}(h(v))$. Therefore by construction of the quandles Q_G and Q'_G, h induces an isomorphism h_Q from Q_G to Q'_G taking (u, i) to $(h(u), i)$. Indeed for vertices u and v in G, we have that $(u, i) \in \varphi_G(v, j)$ holds if and only if $(h(u), i) \in \varphi_{G'}(h(v), j)$ holds. The verification that $x *_G y = z$ if and only if $h_Q(x) *_{G'} h_Q(y) = h_Q(z)$ follows immediately. □

For the converse, we will show that any two isomorphic keis of the form Q_G and $Q_{G'}$ admit an isomorphism induced by an isomorphism of the underlying graphs G and G'. Not all kei isomorphisms between Q_G and $Q_{G'}$ arise from graph isomorphisms; indeed, Lemma 5 gives continuum many others. Also, if the graph K is the complete irreflexive directed graph on V, then Q_K is the trivial kei on $V \times \{0, 1\}$, with $(u, i) * (v, j) = (v, j)$ for all (u, i) and (v, j). Of course there are many automorphisms of the trivial kei that are not of the form given by Lemma 5 or induced by a graph isomorphism: any permutation of the underlying set $V \times \{0, 1\}$ is an automorphism of this kei. We will see in the Claim that follows that any kei isomorphism ρ between folded keis splits into two parts, one of the type described by Lemma 5 and one given by an automorphisms of a trivial kei. Each of these can be converted into a partial isomorphism of the desired form, and the pieces recombined to yield the graph isomorphism required for the Theorem.

To aid with intuition, for any graph $G = (V, E)$ with associated kei $Q_G = (V \times \{0, 1\}, *_G)$, we refer to $V \times \{0\} \subset Q_G$ as the *bottom* of Q_G and $V \times \{1\} \subset Q_G$ as the *top* of Q_G. Also for any v in V we refer to each of $(v, 0)$ and $(v, 1)$ as the *twin* of the other.

Claim. *Suppose $G = (V_G, E_G)$ and $G' = (V_{G'}, E_{G'})$ are irreflexive directed graphs such that there is a kei isomorphism ρ with $\rho\colon Q_G \to Q_{G'}$. Then there is bijection f from $V_G \to V_{G'}$ such that, viewed as a map from G to G', f is a graph isomorphism.*

Proof of Claim. For any graph $H = (V, E)$, we split the underlying set V into two components, which we call the "fixed points" and the "moving points" based on their behaviour in the quandle Q_H. The purely graph-theoretic definitions of the fixed points and moving points is simpler, so we give them first: the fixed points are those which are complete for inward edges, and the moving points are those that are not. That is,

$$F_H = \{v \in V \mid \forall u \in V (u\ E\ v)\}.$$

From the quandle point of view, the fixed points may equivalently be defined as those v for which left multiplication by any element of Q_H does not swap $(v,0)$ with $(v,1)$, that is,

$$F_H = \{v \in V \mid \forall (u,i) \in Q_H[(u,i) *_H (v,0) = (v,0)]\}.$$

The moving points are then those not in F_H, that is, $M_H = V \setminus F_H$.

We shall define the function $f : V_G \to V_{G'}$ piecewise, giving separately the restrictions of f to the fixed points F_G and the moving points M_G. In fact, these restrictions will themselves be bijections from F_G to $F_{G'}$ and from M_G to $M_{G'}$, as is clearly necessary for f to be a graph isomorphism.

We are given an isomorphism $\rho : Q_G \to Q_{G'}$. Let us denote by $\rho_V(v,i)$ and $\rho_I(v,i)$ respectively the first and second components of $\rho(v,i)$: that is, $\rho(v,i) = (\rho_V(v,i), \rho_I(v,i))$.

First we define f on the moving points. If v is in M_G, then there is some (u,i) in Q_G that moves $(v,0)$. That is, the vaule of $(u,i) *_G (v,0)$ is not $(v,0)$, and hence by the definition of $*_G$ it must be that $(u,i) * (v,0)$ is $(v,1)$, and furthermore that $(u,i) * (v,1)$ is $(v,0)$. Applying the kei isomorphism ρ we have that $\rho(u,i) * \rho(v,0) = \rho(v,1)$ holds, and by injectivity $\rho(v,1) \neq \rho(v,0)$. By the definition of $*_{G'}$, the first components of $\rho(v,0)$ and $\rho(v,1)$ must be equal. We take $f(v)$ to be this value: $f(v) = \rho_V(v,0) = \rho_V(v,1)$.

Clearly $f \restriction M_G$ so defined is injective since ρ is a bijection. Moreover $f \restriction M_G$ surjects onto $M_{G'}$. Indeed, for w in $M_{G'}$ and (t,i) in $Q_{G'}$ such that $(t,i) *_{G'} (w,0) \neq (w,0)$, we have $\rho^{-1}(t,i) *_G \rho^{-1}(w,0) \neq \rho^{-1}(w,0)$, and so the first component of $\rho^{-1}(w,0)$ lies in M_G and has image w under f.

To complete the definition of f it remains to give the value of $f(v)$ for those v in F_G. Let v_0 be an element of F_G. Unlike for elements of M_G, it need not be the case that $\rho_V(v_0,0)$ is the same as $\rho_V(v_0,1)$. However, since ρ is surjective, we may find v_1 in F_G and i_{v_1} in $\{0,1\}$ such that $\rho_V(v_1, i_{v_1}) = \rho_V(v_0,1)$ and $\rho_I(v_1,i_{v_1}) = 1 - \rho_I(v_0,1)$: that is, if $\rho(v_0,1)$ is on the bottom of the kei then (v_1,i_{v_1}) is chosen such that $\rho(v_1,i_{v_1})$ is its twin on the top, and conversely if $\rho(v_0,1)$ is on the top of the kei then (v_1,i_{v_1}) is chosen such that $\rho(v_1,i_{v_1})$ is its twin on the bottom. Likewise we may find v_{-1} in F_G and $i_{v_{-1}}$ in $\{0,1\}$ such that $\rho_V(v_{-1}, 1-i_{v_{-1}}) = \rho_V(v_0,0)$ and $\rho_I(v_0,0) = 1 - \rho_I(v_{-1}, 1-i_{v_{-1}})$. We may inductively extend our definitions, obtaining for all k in $\mathbb{Z}$ a vertex v_k in V_G and i_{v_k} in $\{0,1\}$ (with $i_{v_0} = 0$) such that $\rho_V(v_k, 1-i_{v_k}) = \rho_V(v_{k+1}, i_{v_{k+1}})$. Note that if there is some k such that $v_k = v_0$, then i_{v_k} defined in this way will be equal to i_{v_0}, so our notation i_{v_j} gives a well-defined function from vertices v_j in F_G to members of $\{0,1\}$. Indeed, (construing for now i_{v_j} as a function of j rather than v_j) consider the first repetition in the sequence $(v_0,i_{v_0}), (v_0, 1-i_{v_0}), (v_1, i_{v_1}), \ldots$. Clearly if (v_k,i_{v_k}) is distinct from all of its predecessors in the sequence, then so too is $(v_k, 1-i_{v_k})$. Thus, the first repetition in the sequence must be of the form (v_k,i_{v_k}). If $(v_k,i_{v_k}) = (v_j, 1-i_{v_j})$ for some $j < k$, then of course $\rho(v_k,i_{v_k}) = \rho(v_j, 1-i_{v_j})$, so swapping betweeen the top and bottom of the kei, we have from the inductive construction that $\rho(v_{k-1}, 1-i_{v_{k-1}}) = \rho(v_{j+1}, i_{v_{j+1}})$. But then by the minimality of k as giving a repetition, we must have $j = k-1$, so $(v_k,i_{v_k}) = (v_{k-1}, 1-i_{v_{k-1}})$, violating the fact from the construction that $\rho(v_k,i_{v_k}) \neq \rho(v_{k-1}, 1-i_{v_{k-1}})$.

The set $\{v_j \mid j \in \mathbb{Z}\}$ may be finite or infinite, but the corrresponding subset $\{\rho_V(v_j,i_{v_j}) \mid j \in \mathbb{Z}\}$ has the same cardinality: $(v_j,i_{v_j}) = (v_k,i_{v_k})$ if and only if $\rho(v_j,i_{v_j}) = \rho(v_k,i_{v_k})$. Note also that for each k, the left multiplication maps $m_{\rho(v_k,1-i_{v_k})}$ and $m_{\rho(v_{k+1},i_{v_{k+1}})}$ on $Q_{G'}$ are the same since $\rho(v_k, 1-i_{v_k})$ and $\rho(v_{k+1}, i_{v_{k+1}})$ have the same first component. Therefore $m_{(v_k,1-i_{v_k})}$ and $m_{(v_{k+1},i_{v_{k+1}})}$ are the same on Q_G. It follows that v_k and v_{k+1} have outward edges to the same other vertices in G, as well as to each other, and by induction the same is true of all members of the set $\{v_k \mid k \in \mathbb{Z}\}$; likewise, all members of the set $\{\rho_V(v_k, i_{v_k})\}$ have edges to one another and to the same other vertices.

The set F_G may be partitioned into such "cycles" of vertices $\{v_k \mid k \in \mathbb{Z}\}$ by choosing a starting vertex v_0 in each cycle. With such choices made, we in particular have an assignment of i_v in $\{0,1\}$ to each v in F_G, and may define $f \restriction F_G$ by $f(v) = \rho_V(v, i_v)$. Clearly with this definition $f \restriction F_G$ is a bijection from F_G to its image. Moreover its image is all of $F_{G'}$: if $(t,i) *_{G'} (w,0) = (w,0)$ for all (t,i) in $Q_{G'}$, then $\rho^{-1}(t,i) *_G \rho^{-1}(w,0) = \rho^{-1}(w,0)$ for all (t,i) in $Q_{G'}$, that is, $(u,j) *_G \rho^{-1}(w,0) = \rho^{-1}(w,0)$ for all (u,j) in Q_G.

We have thus constructed a bijection $f : V_G \to V_{G'}$, and it remains to show that f is in fact a graph isomorphism from G to G'. So let u and v be vertices of G. If v is in F_G, then $f(v)$ is in $F_{G'}$, so both $u \; E_G \; v$ and $f(u) \; E_{G'} \; f(v)$ hold. Suppose v is in M_G. If u is in F_G we have i_u in $\{0,1\}$ as defined above, and otherwise take $i_u = 0$. Then

$$(u, i_u) *_G (v, 0) = \begin{cases} (v,0) & \text{if } u \; E_G \; v \text{ or } u = v \\ (v,1) & \text{otherwise,} \end{cases}$$

so

$$\rho(u, i_u) *_G \rho(v, 0) = \begin{cases} \rho(v,0) & \text{if } u \; E_G \; v \text{ or } u = v \\ \rho(v,1) & \text{otherwise.} \end{cases}$$

Since the first component of $\rho(u, i_u)$ is $f(u)$ and the first component of $\rho(v, 0)$ is $f(v)$, we have that $f(u) \; E'_G \; f(v)$ if and only if $u \; E_G \; v$, completing the proof that f is a graph isomorphism from G to G'. □

With the Claim we have shown that, whilst not every isomorphism of keis Q_G and $Q_{G'}$ need arise from a graph isomorphism, such an isomorphism can be used to define a graph isomorphism of G and G', which by the Remark gives rise to a (potentially different) isomorphism of Q_G and Q'_G. This completes the proof of Theorem 8. □

Theorem 9. *The classes of keis, quandles, racks, left distributive algebras, and algebras satisfying Σ are each Borel complete.*

Proof. Implicit in the statement that these classes of structures are Borel complete is that we are considering the classes of countable such structures with underlying set $\mathbb{N}$, with each class topologized as described in Section 2.

The map $G \mapsto Q_G$ from the class of graphs to the class of keis is not only Borel but in fact continuous. Recall from Section 2 that the subbasic open sets in the space of graphs are of the form either $\{G \mid m \; E \; n\}$ or $\{G \mid m \not\!E \; n\}$. Similarly, for quandles with underlying set $\mathbb{N}$, the subbasic open sets are of the form $\{(\mathbb{N}, *) \mid u * v = w\}$ or $\{(\mathbb{N}, *) \mid u * v \neq w\}$. Then by the construction of our dynamical keis, it is clear that the inverse image of any open set is open (as we defined $*$ in terms of the edge relation of E). Hence the map taking G to Q_G is continuous and so certainly Borel, and therefore because the class of graphs is Borel complete, we have shown that the keis, and hence quandles, and hence racks, and hence left distributive algebras are Borel complete.

Because the language of Σ is different from that of left distributive algebras, a different argument is needed to show that the class of algebras satisfying Σ is Borel complete. For this we utilize the result of Mekler [18] that the class of groups is Borel complete (see [9, §2.3] for a sketch of the argument). As discussed after Definition 2, every group endowed with its conjugation operation and its group operation satisfies Σ. The inclusion map $(G, \circ) \mapsto (G, \circ, *)$ where $\circ$ denotes the group operation and $*$ denotes conjugation is easily seen to be continuous and so is certainly Borel. Of course, since the group operation is one of the two operations in the language of Σ, and the other is conjugation which is determined by the group operation, two groups are isomorphic if and only if their corresponding structures satisfying Σ are isomorphic. We thus have that group isomorphism Borel reduces to isomorphism as algebras satisfying Σ, and therefore that the latter is Borel complete. $\square$

4 Concluding remarks

We have shown that in the Borel reducibility sense, the class of left distributive algebras is as complex as possible. Another formalization of the question of complexity is in a category-theoretic setting. Just as the class of graphs is maximal in the Borel completeness sense (and indeed our proof made use of this fact), the *category* of graphs is universal in the sense that every algebraic category fully embeds into it [20, Theorem 5.3]. There are many such universality results for other categories — see, for example, [20] — raising the following natural question.

Question. *Does the category of graphs fully embed into the category of left distributive algebras?*

Of course the same question may also be asked of the category of racks, the category of quandles, and the category of keis. We note that the construction of Q_G from G in Theorem 8 is not even functorial, since a graph homomorphism need not preserve non-edges. Potentially an even more problematic obstacle, however, is the fullness requirement — we have seen that dynamical keis admit many more homomorphisms than simply those arising from graph homomorphisms, at least in our construction. On the other hand, even if it turns out that the category of graphs cannot be fully embedded into the category of keis because keis always admit many homomorphisms, there may be interesting minimal-non-fullness, maximal-complexity results to be obtained in this direction. As an analogy, there can be no full embedding of the category of graphs into the category of abelian groups, as any two abelian groups A and B admit at least one homomorphism between them (the 0 map) and the set of homomorphisms between them $\mathrm{Hom}(A, B)$ naturally forms an abelian group. Nevertheless Przeździecki [19] has shown that there is an embedding $\mathcal{A}$ from the category of graphs to the category of abelian groups such that $\mathrm{Hom}(\mathcal{A}G, \mathcal{A}G')$ is the free abelian group generated by $\mathrm{Hom}(G, G')$ — the best possible result given these constraints.

As mentioned in the the introduction, the implication operation in a Boolean Algebra is left distributive. Borel completeness of the isomorphism relation on Boolean algebras was proven by Camerlo and Gao in [3] and does not follow from what we have proven. Their work shows that a classification of countable Boolean algebras due to Ketonen uses objects for the complete invariants that "cannot be improved in an essential way" [3].

In contrast, our main result is that the class of quandles is Borel complete while tame knots are trivial in terms of Borel reducibility. Whilst the subclass of finitely presented quandles contains the quandles associated with all tame knots and is itself trivial in this context, it is not clear that this finite presentability constraint can be used in practice to simplify the quandle isomorphism problem. Thus, our result suggests that there may well exist a more practical complete invariant for tame knots, with an isomorphism problem that is not as difficult as that for quandles.

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