TWO STEPS TRANSPORTATION PROBLEM

By

Ezio Marchi

Emeritus Professor IMASL
Universidad Nacional de San Luis, San Luis, Argentina
e-mail: emarchi1940@gmail.com

Let $i:1,...,m$ be origins and $k:1,...,p$ be destinations. The merchandise, which is assumed to be indistinguishable, goes from an origin to a destination. However, the merchandise leaving an origin goes through a deposit $j:1,...,n$ and reaches a destination. Each origin i has a capacity r_i and each destination k needs the amount t_k. We have the common condition

$$\sum_{i=1}^{m} r_i = \sum_{k=1}^{p} t_k = r$$

which is concerned with the fact that all the merchandise required is distributed. Such condition is natural in transportation problems. Thus, if $x_{ij}^{1} \geq 0$ and $x_{jk}^{2} \geq 0$ are the respective total amounts transported from origin i to the deposit j, and from there to destination k, then the two-step transportation problem can take the following expression:

$$\begin{aligned} &\sum_{j=1}^{m} x_{ij}^{1} = r_j && i \in \{1,...,m\} = I \\ &\sum_{j=1}^{m} x_{jk}^{2} = t_k && k \in \{1,...,p\} = K \qquad (1,1) \\ &\sum_{i=1}^{m} x_{ij}^{i} - \sum_{k=1}^{p} x_{jk}^{2} = 0 && j \in \{1,...,n\} = J \end{aligned}$$

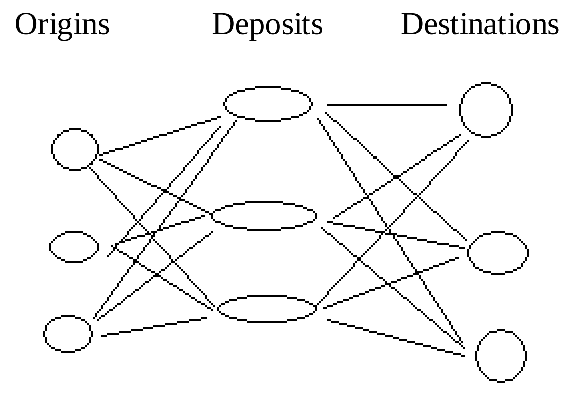
with $x = (x^1, x^2) \geq 0$

The last equation expresses the fact that at each deposits all the incoming amounts go out.

The conditions above are concerned with the total transported amounts, but the complete transportation problem is related to a cost function

$$\begin{aligned} f(x) &= c^1 x^1 + c^2 x^2 = \min! \qquad (1,2) \\ &= \sum_{ij} c_{ij}^{1} x_{ij}^{1} + \sum_{jk} c_{jk}^{2} x_{jk}^{2} = \min! \end{aligned}$$

which is linear, where $x^1 = \{x_{ij}^{1}\}$ and $x^2 = \{x_{ij}^{2}\}$. The amounts c_{ij}^{1} and c_{jk}^{2} are the costs to carry the unit amount from i to j and from j to k, respectively. A solution of (1,1) and (1,2) is called feasible.



It is possible to arrange the linear system (1,1) in a matrix form $Ax = b$, where A is:

1………1	*1………1*	. . . .	*1………1*					↓ m ↑
				1………1	*1………1*	. . . .	*1………1*	↓ p ↑
1 . . . *1*	*1* . . . *1*	. . . . .	*1* . . . *1*	*−1* . . . *−1*	*−1* . . . *−1*	. . . . .	*−1* . . . *−1*	↓ n ↑
$\xleftrightarrow{mn}$					$\xleftrightarrow{pn}$			

and $b = (r_j, t_k, 0)^t$. The rank of matrix A is $m + p + n - 1$.

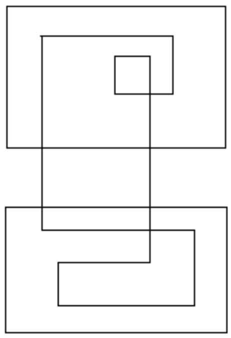
Definition: The support of $s = (y^1_{ij}, y^2_{jk})$ is the set $S(s) = \{(i,j)/ y^1_{ij} > 0\} \cup \{(j,k)/ y^2_{jk} > 0\}$

Definition: Let $s = (y^1_{ij}, y^2_{jk})$ a solution of the 2-step transportation problem, a cycle in the support of s is a sequence of elements of S(s), such that:

$$\sum_j y^1_{ij} = 0 \qquad \forall i$$

$$\sum_j y^2_{jk} = 0 \qquad \forall k$$

$$\sum_i y^1_{ij} = \sum_k y^2_{jk} \qquad \forall j$$



Characterization of extremals

Theorem: A factible solution c of the 2-step transportation problem defined above is extremal if and only if it has no cycles in its support S(c).

Theorem: Each extremal of the problem has at most $m + p + n - 1$ positive components.

Algorithm: Take $(\bar{i}, \bar{j}, \bar{k})$ and consider $a_1 = \min(r_{\bar{i}}, t_{\bar{k}}) = \bar{x}^1_{ij} = \bar{x}^2_{jk}$. If $a_1 = r_{\bar{i}}$, delete the row $\bar{i}$ and take the problem in $I - \{\bar{i}\}, J, K$. The new problem has the entries r_i for $i \neq \bar{i}$, and $t_{\bar{k}} - r_{\bar{i}} \geq 0$ for $k = \bar{k}$ and t_k for $k \neq \bar{k}$. Follow as in the first step, in this way adding up all the entries in the different steps in the respective places it converges to a solution of (1,1). We call it the algorithm.

Theorem: The algorithm converges and provides all the extremals.

Example: Consider a 2-steps problem with 2 origins, 3 deposits and 2 destinations with matricial form:

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
 & & & & & & & & & & & \\
1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x^{1}_{11} \\ x^{1}_{12} \\ x^{1}_{13} \\ x^{1}_{21} \\ x^{1}_{22} \\ x^{1}_{23} \\ x^{2}_{11} \\ x^{2}_{21} \\ x^{2}_{31} \\ x^{2}_{12} \\ x^{2}_{22} \\ x^{2}_{32}
\end{bmatrix}
=
\begin{bmatrix}
1 \\ 5 \\ 3 \\ 3 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

which it has as solutions:

solution 1:

1	
2	3
3	
	3

solution 2:

0.5	0.5
2.5	2.5
1.5	1.5
1.5	1.5

Solution 1 is extremal, solution 2 it is not because is a convex combination of solutions:

1	
5	
3	
3	

	1
	5
	3
	3

The two steps transportation problem cannot be formulated as a particular case of the classic transportation problem. Results with deposit capacities are also obtained.

It is remarkable that the two steps transportation p`roblem cannot be reduced to a classic transportation problem.

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