A Candidate for the Generalised Real Line

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Abstract. Let κ be an uncountable cardinal with $\kappa^{<\kappa} = \kappa$. In this paper we introduce $\mathbb{R}_\kappa$, a Cauchy-complete real closed field of cardinality 2^κ. We will prove that $\mathbb{R}_\kappa$ shares many features with $\mathbb{R}$ which have a key role in real analysis and computable analysis. In particular, we will prove that the Intermediate Value Theorem holds for a non-trivial subclass of continuous functions over $\mathbb{R}_\kappa$. We propose $\mathbb{R}_\kappa$ as a candidate for extending computable analysis to generalised Baire spaces.

1 Introduction

Computable analysis is the study of the computational properties of real analysis. We refer the reader to [21] for an introduction to computable analysis. In classical computability theory one studies the computational properties of functions over natural numbers and transfers these properties to arbitrary countable spaces via coding. The same approach is taken in computable analysis. By using coding, in fact, one can transfer topological and computational results from the Baire space ω^ω to sets of cardinality $2^{\aleph_0}$. In particular, by encoding the real numbers, one can use the Baire space to study computability in the context of real analysis.

Of particular interest in computable analysis is the study of the computational content of theorems from classical analysis. The idea is that of formalizing the complexity of theorems by means similar to those used in computability theory to classify functions over the natural numbers. In this context, the *Weihrauch* theory of reducibility plays an important role. For an introduction to the theory of Weihrauch reductions, see [4]. Weihrauch reductions can be used to classify functions over the Baire space ω^ω. By using this concept it is possible to arrange many theorems from classical real analysis in a complexity hierarchy called the Weihrauch hierarchy. A study of the Weihrauch degrees of some of the most important theorems from real analysis can be found in [4] and [3].

Recently, the study of the descriptive set theory of the generalised Baire spaces κ^κ for cardinals $\kappa > \omega$ has been catching the interest of set theorists (see [12] for an overview on the subject). This fact is also witnessed by the increasing

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number of workshops dedicated to generalised Baire spaces organized in the last two years (AST 2014 in Amsterdam and a satellite workshop to DMV 2015 in Hamburg). Even though generalised Baire spaces are not a new concept in set theory, many aspects of this theory are still unknown. In particular there has been no attempt to generalise computable analysis to spaces of cardinality 2^κ.

This paper provides the foundational basis for the study of *generalised computable analysis*, namely the generalisation of computable analysis to generalised Baire spaces. Since in classical computable analysis and classical Weihrauch theory the field of real numbers has a central role, a question arises naturally in this context: what is the right generalisation of $\mathbb{R}$ in the context of generalised computable analysis?

$$\begin{tikzcd}
\mathbb{R} \arrow[d, dashed, "\text{Generalisation}"'] & \omega^\omega \arrow[l, "\text{Coding}"'] \arrow[d, dashed, "\text{Generalisation}"] \\
? & \kappa^\kappa \arrow[l, "\text{Coding}"']
\end{tikzcd}$$

In this paper we answer this question. In particular, we will introduce $\mathbb{R}_\kappa$, a generalisation of the real line, which provides a well behaved environment for generalising classical results from real analysis to uncountable cardinals. We propose $\mathbb{R}_\kappa$ as the starting point for the study of generalised computable analysis.

The problem of generalising the real line is not new in mathematics. Different approaches have been tried for very different purposes. A good introduction to these number systems can be found in [9]. Among the most influential contributions to this field particularly important are the works of Sikorski [20] and Klaua [15] on the *real ordinal numbers* and that of Conway [6] on the *surreal numbers*. Sikorski's idea was to repeat the classical Dedekind construction of the real numbers starting from an ordinal equipped with the Hessenberg operations (i.e., commutative operations over the ordinal numbers). Unfortunately, one can prove that these fields do not have the density properties that, as we will see, will have a central role in the context of real analysis. The surreal numbers were introduced by Conway in order to generalise both the Dedekind construction of real numbers and the Cantor construction of ordinal numbers. In his introduction to surreal numbers, Conway proved that they form a (class) real closed field. Later, Ehrlich [15] proved that every real closed field is isomorphic to a subfield of the surreal numbers, showing that they behave like a universal (class) model for real closed fields. It is then natural for us to use this framework in the developing of $\mathbb{R}_\kappa$.

As we will see, doing analysis over field extensions of $\mathbb{R}$ is not an easy task. This is due to the fact that no proper ordered field extension of $\mathbb{R}$ is connected. However, many of the basic theorems of real analysis are linked to the fact that $\mathbb{R}$ is a connected space. To overcome this problem, instead of using standard topological tools, we will use a different mathematical framework which, under specific conditions over the density of $\mathbb{R}_\kappa$, will allow us to see our field extension of $\mathbb{R}$ as a linear continuum.

In this paper, we shall give necessary requirements for a space $\mathbb{R}_\kappa$ to be the generalisation of the real numbers. Considering the Intermediate Value Theorem (IVT) as one of the pillars of real analysis, we place particular emphasis on its validity in the generalised case, and we develop the requirements in such a way that they will allow us to prove it.

2 The surreal numbers

In this paper κ will refer to a fixed cardinal larger than ω. As usual in generalised descriptive set theory, let kappa be an uncountable cardinal with $\kappa^{<\kappa} = \kappa$. Note in particular that this assumption implies that κ is a regular cardinal.

We will assume basic knowledge of topology, field theory and computable analysis. A good introduction to these subjects can be found in [18], [5] and [21], respectively.

The following definition as well as most of the results in this section, are due to Conway [6] and have also been deeply studied by Gonshor in [14].

Definition 1 (Surreal numbers). *A* surreal number *is a function from an ordinal $\alpha \in \mathrm{On}$ to $\{+,-\}$, i.e., a sequence of pluses and minuses of ordinal length. We will denote the class of surreal numbers by* No. *The* length *of a surreal number $x \in \mathrm{No}$ is the smallest ordinal $\ell(x) \in \mathrm{On}$ for which x is not defined.*

We can define a total order over No as follows:

Definition 2. *Let $x, y \in \mathrm{No}$ be two surreal numbers. We say that x is smaller than y in symbol $x < y$ iff $x(\alpha) < y(\alpha)$, where α is the smallest ordinal s.t. $x(\alpha) \neq y(\alpha)$ using the order $- < 0 < +$ where $x(\alpha) = 0$ if x is not defined at α.*

According to Conway's original idea, every surreal number is generated by filling some cut between shorter numbers. The following theorem gives us a connection between this intuition and the surreal numbers as we have defined them.

Theorem 3 (Simplicity theorem). *Let L and R be two sets of surreal numbers such that $L < R$. Then there is a unique surreal z, denoted by $[L|R]$, of minimal length such that $L < \{z\} < R$. We will call $[L|R]$ a representation of z.*

By using the Simplicity Theorem Conway defined the field operations over No and proved that these operations satisfy the axioms of real closed fields. Later Ehrlich proved that the class field No behaves like a universal model for the theory of real closed fields, this means that every set-like model of the theory of real closed fields is isomorphic to a subfield of No. In particular Conway proved that the real numbers are a subfield of $\mathrm{No}_{\leq\omega}$.

The last theorem we want to mention in this section is due to van der Dries and Ehrlich [8]:

Theorem 4 (van der Dries & Ehrlich). *The set of surreal numbers $\mathrm{No}_{<\kappa}$ is a real closed field.*

3 Super dense κ-real extensions of $\mathbb{R}$

In this paper we will have a quasi-axiomatic approach. In particular, we will first determine the properties that we need on $\mathbb{R}_\kappa$ in order to prove some basic facts from classical analysis. Then we will show how it is possible to define $\mathbb{R}_\kappa$ as a subfield of the surreal numbers.

Let us consider some of the basic properties that we expect from $\mathbb{R}_\kappa$. First of all we want $\mathbb{R}_\kappa$ to be a generalisation of $\mathbb{R}$ to the uncountable cardinal κ. Therefore we require that $\mathbb{R}_\kappa$ is a proper ordered field extension of $\mathbb{R}$. As we said, we want to use $\mathbb{R}_\kappa$ to do analysis. For this reason, we expect $\mathbb{R}_\kappa$ to behave as much as possible like $\mathbb{R}$. Formally we will require that $\mathbb{R}_\kappa$ is a real closed field extending $\mathbb{R}$. Since the theory of real closed fields is complete [17, Corollary 3.3.16], this implies that $\mathbb{R}_\kappa$ has the same first order properties as $\mathbb{R}$[1].

REQUIREMENT R1: $\mathbb{R}_\kappa$ is a real closed field extending $\mathbb{R}$.

Now, since we want to use $\mathbb{R}_\kappa$ to do computable analysis over sets of cardinality 2^κ, we require that $|\mathbb{R}_\kappa| = 2^\kappa$.

REQUIREMENT R2: $\mathbb{R}_\kappa$ has cardinality 2^κ.

Finally, since the set of rational numbers $\mathbb{Q}$ has a central role in the representation theory of $\mathbb{R}$ (the interested reader is referred to [21]), we want $\mathbb{R}_\kappa$ to have a dense subset which can play the same role as $\mathbb{Q}$.

REQUIREMENT R3: $\mathbb{R}_\kappa$ has a dense subset of cardinality κ.

In general we define:

Definition 5 (κ-real extension of $\mathbb{R}$). *Let K be a field satisfying* R1*,* R2*,* R3*. Then we will call K a* κ-real extension of $\mathbb{R}$*.*

Now we focus on those properties that are needed to extend theorems from classical analysis to $\mathbb{R}_\kappa$. Many of these classical results depend on the order over $\mathbb{R}$ and on its interval topology. So we will start considering interval topologies over κ-real extensions of $\mathbb{R}$ and their properties. First we recall few facts from field theory and classical analysis. It is a well known result from classical analysis that $\mathbb{R}$ has no Dedekind complete ordered field extensions (see [5, Corollary 8.7.4]). Therefore $\mathbb{R}_\kappa$ will not be Dedekind complete. More generally we have:

Corollary 6. *Let K be a κ-real extension of $\mathbb{R}$. Then K is not Dedekind complete.*

As usual, given an ordered field K, one can define Cauchy sequences over K.

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[1] In this paper we will use gray boxes for Requirements. Requirements are properties that the definition of $\mathbb{R}_\kappa$ will have to satisfy.

Definition 7 (Cauchy sequences). *Let α be an ordinal and K^+ the positive part of K. Then a sequence $(x_i)_{i\in\alpha}$ of elements of K is* Cauchy *iff*

$$\forall\varepsilon \in K^+ \exists\beta < \alpha \forall\gamma, \gamma' \geq \beta.\ |x_{\gamma'} - x_\gamma| < \varepsilon.$$

The sequence is convergent *if there is $x \in G$ such that*

$$\forall\varepsilon \in K^+ \exists\beta < \alpha \forall\gamma \geq \beta.\ |x_\gamma - x| < \varepsilon.$$

We will call x the limit *of $(x_i)_{i\in\alpha}$. Given a group K it is said to be Cauchy complete iff every Cauchy sequence whose length is equal to the degree* $\mathrm{Deg}(K)$ *of K has a limit in K.*

It is a an easy exercise to see that Cauchy and Dedekind completeness coincide on Archimedean fields, while on non-Archimedean fields Cauchy completeness is weaker than Dedekind completeness.

Another property which is central in mathematical analysis is connectedness. It turns out that connectedness and Dedekind completeness are equivalent properties. Therefore, it is easy to see that we will not be able to define $\mathbb{R}_\kappa$ in such a way that its interval topology is connected.

As we said, our main purpose is that of proving basic facts from analysis over $\mathbb{R}_\kappa$. In particular we want to be able to prove the Intermediate Value Theorem (IVT). It turns out that if the IVT holds on an ordered field K then K is connected. This means that we cannot aim to prove the IVT over κ-real extensions of $\mathbb{R}$ in all its strength.

3.1 κ-topologies

Given what we have proved in the previous section, it is hard to do analysis over κ-real extensions of $\mathbb{R}$ by using standard topological tools. To overcome this problem we will use a tool introduced by Alling called κ-topology. A similar approach to do analysis over the surreal numbers was taken in [19].

Definition 8 (κ-topology). *A κ-*topology *τ over a set X is a collection of subsets of X such that:*

1. *$\emptyset, X \in \tau$.*
2. *$\forall\alpha < \kappa$. if $\{A_i\}_{i\in\alpha}$ is a collection of sets in τ, then $\bigcup_{i<\alpha} A_i \in \tau$.*
3. *$\forall A, B \in \tau$. $A \cap B \in \tau$.*

*The elements of τ are called κ-*open *sets.*

Intuitively, the reason why we use κ-topologies is that, as we have seen in the previous section, interval topologies over κ-real extensions of $\mathbb{R}$ are too fine. As we will see κ-topologies will be coarser than topologies and will allow us to prove a weaker version of the Intermediate Value Theorem over particularly well-behaved κ-real extensions of $\mathbb{R}$.

Theorem 9 (Alling). *Let X be a set and B be a topological base over X. Then the set τ_κ defined as follows: $\emptyset, X \in \tau_\kappa$ and union of less than κ elements of B is in τ_κ, is a κ-topology. We will call τ_κ the κ-topology generated by B. Moreover we will call B a base for the κ-topology.*

Obviously many topological definitions can be relativized to κ-topologies. In particular we have the following:

Definition 10 (κ-continuity). *Let X and Y be two sets and τ, τ' be two κ-topologies respectively on X and on Y. Then $f : X \to Y$ is a* κ-continuous *function iff $\forall U \in \tau'.\ f^{-1}[U] \in \tau$.*

Definition 11 (κ-connectedness). *Let X be a set and τ be a κ-topology over X. Then X is* κ-connected *iff $\forall U, V \in \tau.\ X = U \cup V \wedge U \cap V = \emptyset \Rightarrow U = \emptyset \vee V = \emptyset$.*

All these definitions behave quite well with respect to their topological counterparts. Indeed, many classical theorems from topology hold for κ-topologies (see [1]). However, there are theorems from topology that are not valid on κ-topologies. Typically for κ-topologies local properties do no transfer to global properties (e.g. in κ-topologies openness is not implied by local openness).

Now we will introduce a κ-topological analogous of the interval topology over an ordered set.

Definition 12 (Interval κ-topology). *Let X be an ordered set and B be the set of open intervals with end points in $X \cup \{+\infty, -\infty\}$. We will call* interval κ-topology *over X the κ-topology generated by B.*

From now on we will consider the interval κ-topology as the standard κ-topology over κ-real extensions of $\mathbb{R}$.

As we have seen, in order to be able to prove some basic theorems from analysis we need to work within a connected space. However, as we have already pointed out, we can not aim for connectedness of κ-real extensions of $\mathbb{R}$. The next result is due to Alling [1] and it makes precise the connection between the density of an ordered set and the connectedness of its interval κ-topology.

Definition 13 (Hausdorff η_κ-set). *Let X be an ordered set and κ be a cardinal. We say that X is an* η_κ-set *iff given $L, R \subseteq X$, such that $L < R$ and $|L| + |R| < \kappa$ then there is $x \in X$ such that $L < \{x\} < R$.*

Theorem 14 (Alling). *Let X be an η_κ-set endowed with the interval κ-topology and X' a subset of X. Then X' is κ-connected iff X' is an interval in X.*

In view of Theorem 14, it is natural to require:

REQUIREMENT R4: $\mathbb{R}_\kappa$ is an η_κ-set.

Definition 15. *A field $K \supseteq \mathbb{R}$ is called a* super dense κ-real extension of $\mathbb{R}$ *if it satisfies requirements R1, R2, R3, and R4.*

As in classical topology κ-continuous functions preserve κ-connectedness.

Theorem 16. *Let $f : X \to Y$ be a κ-continuous function. If X is κ-connected then $f(X)$ is κ-connected.*

3.2 Analysis over super dense κ-real extensions of $\mathbb{R}$

Using the results from the previous section we can modify the standard topological proof of the IVT to show that its restriction to κ-continuous functions holds over super dense κ-real extensions of $\mathbb{R}$.

Theorem 17 ($\mathrm{IVT}_{\kappa}^{K}$). *Let K be a super dense κ-real extension of $\mathbb{R}$, the set $[a, b] \subset K$ be a closed subinterval of K and $f : [a, b] \to K$ be a κ-continuous function. Then for every $r \in K$ such that r is in between $f(a)$ and $f(b)$, there is $c \in [a, b]$ such that $f(c) = r$.*

It is a well-known fact that in every real closed field the IVT holds for polynomials in one variable (see [17, Theorem 3.3.9]), therefore it is natural to ask if polynomials over super dense κ-real extensions of $\mathbb{R}$ are κ-continuous.

Theorem 18. *Let K be a super dense κ-real extension of $\mathbb{R}$ and p be a polynomial in one variable with coefficients in K. Then p is κ-continuous.*

4 The generalised real line $\mathbb{R}_{\kappa}$

We are now ready to define $\mathbb{R}_{\kappa}$. A naïve attempt to define such extension would be that of starting from κ endowed with the surreal operations (i.e., the Hessenberg operations) and try to repeat the standard construction of $\mathbb{Z}^{\kappa}$ and $\mathbb{Q}^{\kappa}$. Then, we could define $\mathbb{R}^{\kappa}$ as the Cauchy completion of $\mathbb{Q}^{\kappa}$ obtaining a Cauchy complete field. Unfortunately this approach does not work. This is due to the fact that, as Sikorski proved, the field $\mathbb{Q}^{\kappa}$ is Cauchy complete and then $\mathbb{R}^{\kappa} = \mathbb{Q}^{\kappa}$. Recall that $\mathbb{Q}^{\kappa}$ is a set of equivalence classes of pairs of elements in $\mathbb{Z}^{\kappa}$, hence it has cardinality at most κ. Therefore $\mathbb{R}^{\kappa}$ violates R2 and is not a good candidate for our purposes. This construction appeared for the first time in a paper from Sikorski in 1948 [20] (see also [2] and [15] for a complete study of this approach). For this reason we have to take a different approach in defining $\mathbb{R}_{\kappa}$.

By Theorem 4 we know that $\mathrm{No}_{<\kappa}$ is a real closed field. Moreover, since $\kappa > \omega$, it is easy to see that $\mathbb{R} \subset \mathrm{No}_{<\kappa}$. In particular this means that R1 holds for $\mathrm{No}_{<\kappa}$. Furthermore, it is not hard to prove that $\mathrm{No}_{<\kappa}$ also satisfies R4. Then we have:

Proposition 19. *The field $\mathrm{No}_{<\kappa}$ has the following properties:*

1. $|\mathrm{No}_{<\kappa}| = \kappa$ *and* $\mathrm{Deg}(\mathrm{No}_{<\kappa}) = \kappa$.
2. $\mathrm{Cof}(\mathrm{No}_{<\kappa}) = \mathrm{Coi}(\mathrm{No}_{<\kappa}) = \kappa$ *and* $\mathrm{No}_{<\kappa}$ *has weight* κ.

Proposition 19 tells us that $\mathrm{No}_{<\kappa}$ has almost all the properties that we want from $\mathbb{R}_{\kappa}$ but is still too small. Moreover, it is not hard to see that $\mathrm{No}_{<\kappa}$ is not Cauchy complete in the sense of Definition 7 this fact is particularly problematic in the context of computable analysis, where most of the classical representations of $\mathbb{R}$ rely on the fact that $\mathbb{R}$ is the Cauchy completion of $\mathbb{Q}$.

It is therefore natural to consider $\mathrm{No}_{<\kappa}$ as generalised rational numbers, and to define $\mathbb{R}_\kappa$ as the Cauchy completion of $\mathrm{No}_{<}$ as in classical analysis[2]. Since we are working within the surreal numbers, this can be done in a natural way.

Definition 20 (Veronese cuts). *Let K be an ordered field. We call $\langle L, R\rangle$ a* cut *over K iff $L, R \subseteq K$ and $L < R$. Moreover we will say that $\langle L, R\rangle$ is a* Veronese cut *iff it is a cut such that, L has no maximum, R has no minimum and for each $\varepsilon \in K^+$ there are $\ell \in L$ and $r \in R$ such that $r < \ell + \varepsilon$. We will say that K is* Veronese complete *iff for each Veronese cut $\langle L, R\rangle$, there is $x \in K$ such that $L < \{x\} < R$.*

It is a well known fact that Cauchy and Veronese completeness are equivalent notions (see [7, 10]). For this reason we can define the Cauchy completion of $\mathrm{No}_{<\kappa}$ by using the Simplicity Theorem as follows:

Definition 21 ($\mathbb{R}_\kappa$). *We define $\mathbb{R}_\kappa$ as follows:*

$$\mathbb{R}_\kappa = \mathrm{No}_{<\kappa} \cup \{x \mid x = [L|R] \text{ where } \langle L, R\rangle \text{ is a Veronese cut on } \mathrm{No}_{<\kappa}\}.$$

Now we will show that $\mathbb{R}_\kappa$ is a super dense κ-real extension of $\mathbb{R}$. First of all we will prove that $\mathrm{No}_{<\kappa}$ is a dense subfield of $\mathbb{R}_\kappa$ and that $\mathbb{R}_\kappa$ is Cauchy complete (i.e., $\mathbb{R}_\kappa$ the Cauchy completion of $\mathrm{No}_{<\kappa}$).

Lemma 22. *The field $\mathrm{No}_{<\kappa}$ is dense in $\mathbb{R}_\kappa$. Moreover the set $\mathbb{R}_\kappa$ is Cauchy complete.*

In view of the previous theorem from now on we will call $\mathrm{No}_{<\kappa}$ the κ-rational numbers and we will use the symbol $\mathbb{Q}_\kappa$ instead of $\mathrm{No}_{<\kappa}$.

Since we have showed that $\mathbb{R}_\kappa$ is the Cauchy completion of a real closed field, by a standard model theoretical argument we have:

Corollary 23. *The set $\mathbb{R}_\kappa$ is a real closed field extending $\mathbb{R}$.*

Now that we have shown that $\mathbb{R}_\kappa$ is a real closed field extending $\mathbb{R}$ we want to check that all the other properties of super dense κ-real extensions of $\mathbb{R}$ hold for $\mathbb{R}_\kappa$.

Theorem 24. *The real closed field $\mathbb{R}_\kappa$ has the following properties:*

1. *$|\mathbb{R}_\kappa| = 2^\kappa$, $\mathrm{Deg}(\mathbb{R}_\kappa) = \kappa$ and $\mathbb{R}_\kappa$ is an η_κ-set.*
2. *$\mathrm{Cof}(\mathbb{R}_\kappa) = \mathrm{Coi}(\mathbb{R}_\kappa) = \kappa$ and the weight of $\mathbb{R}_\kappa$ is κ.*

Proof. We will only prove $|\mathbb{R}_\kappa| = 2^\kappa$ the rest follows from the fact that $\mathbb{Q}_\kappa$ is dense in $\mathbb{R}_\kappa$. We want to prove $2^\kappa \leq |\mathbb{R}_\kappa| \leq 2^\kappa$. On the one hand we have that $\mathbb{R}_\kappa \subset \mathrm{No}_{\leq\kappa}$. Indeed, $\mathrm{No}_{\leq\kappa}$ contains the Dedekind completion of $\mathrm{No}_{<\kappa}$, hence also its Cauchy completion $\mathbb{R}_\kappa$. Then, since $|\mathrm{No}_{\leq\kappa}| = 2^\kappa$, we have that $|\mathbb{R}_\kappa| \leq 2^\kappa$.

On the other hand let $\{0,1\}^{<\kappa}$ be the full binary tree of height κ, we define a tree T which is in bijection with $\{0,1\}^{<\kappa}$ and whose nodes are subintervals of

[2] Note that this also reflects the fact that $\mathrm{No}_{<\omega}$ are the dyadic numbers and $\mathbb{R}$ is the Cauchy completion of $\mathrm{No}_{<\omega}$.

$\mathbb{R}_\kappa$ and whose branches corresponds to different elements of $\mathbb{R}_\kappa$. We define the tree by recursion as follows: set $T_\lambda = (0,1)$ as the root of the tree. Now assume that for $p \in 2^{<\kappa}$ and that the element $T_p \neq \emptyset$ is already defined. We define T_{p0} and T_{p1} as two non-empty disjoint subintervals of T_p such that $T_{p0} = (a_{p0}, b_{p0})$ and $T_{p1} = (a_{p1}, b_{p1})$, where $a_{p0}, b_{p0}, a_{p1}, b_{p1} \in \mathbb{Q}_\kappa$ with $|a_{p0} - b_{p0}| \leq \frac{1}{\ell(p)+1}$ and $|a_{p1} - b_{p1}| \leq \frac{1}{\ell(p)+1}$. Finally if $p \in 2^{<\kappa}$ is of limit length γ and $T_{p\restriction\alpha}$ has already been defined for every $\alpha < \gamma$, we define $T'_p = \bigcap_{\alpha<\gamma} T_{p\restriction\alpha}$. Note that by the fact that $\mathbb{R}_\kappa$ is an η_κ-set, the set T'_p non-empty, moreover T'_p is trivially convex (i.e., if $x, y \in T'_p$ and $x \leq z \leq y$, then $z \in T'_p$). Therefore we can define T_p as we have done for the successor stage starting from T'_p. It follows trivially by the way in which we have defined the tree that the set $\bigcap_{\alpha\in\kappa} T_{p\restriction\alpha}$ contains a single element of $\mathbb{R}_\kappa$. Indeed by the properties of the tree we have that $[\{a_{p\restriction\alpha} \mid \alpha \in \kappa\} \mid \{b_{p\restriction\alpha} \mid \alpha \in \kappa\}]$ is a Veronese cut in $\mathbb{Q}_\kappa$. Therefore $\bigcap_{\alpha\in\kappa} T_{p\restriction\alpha}$ is a singleton in $\mathbb{R}_\kappa$ as desired. Therefore we have $2^\kappa \leq |\mathbb{R}_\kappa|$ as desired.

5 Conclusions & future work

In this paper we have introduced a real closed field extending $\mathbb{R}$ suitable for doing real analysis over the generalised Baire space κ^κ. We have showed that, although it has some limitations intrinsic to the problem, $\mathbb{R}_\kappa$ preserves many interesting properties of the real numbers. In particular we showed:

1. $\mathbb{R}_\kappa$ is a Cauchy complete super dense κ-real extension of $\mathbb{R}$ of cardinality 2^κ.
2. $\mathbb{R}_\kappa$ has a dense subset of cardinality κ and $\mathrm{Coi}(\mathbb{R}^+_\kappa) = \kappa$.
3. The IVT holds for κ-continuous functions.

As we have seen, most of these properties are motivated by computable analysis. For this reason we propose $\mathbb{R}_\kappa$ as the generalised real line in the context of computable analysis. An example of how $\mathbb{R}_\kappa$ can be used to study the topological Weihrauch complexity of theorems of analysis can be found in [13].

In this paper we didn't investigate the uniqueness of $\mathbb{R}_\kappa$. For $\kappa = \aleph_1$, CH holds and $\mathbb{R}_\kappa$ is isomorphic to the unique field that Dales and Woodin call $\widehat{\mathbf{R}}$ (see [7, Theorem 3.21(iv)]). The question is still open for $\kappa > \aleph_1$.

There are two natural continuations of this paper. On one hand it is natural to ask for a study of the computational strength of generalisations of theorems from real analysis. To accomplish this, a theory of generalised type two computability is needed. As we shall show in a paper soon to appear, it is possible to modify the notions of Ordinal Turing Machine introduced by Koepke in [16] to define a generalised version of Type Two Turing Machine (T2TM). The intuition behind this notion is that generalised T2TMs should run classical programs for Turing machines for κ steps instead of just ω. These machines lead to a very natural notion of computability, in which, because of the properties of $\mathbb{R}_\kappa$ and $\mathbb{Q}_\kappa$, the field operations restricted to $\mathbb{Q}_\kappa$ are computable in less than κ steps, while one may need to run forever (i.e., up to κ) to compute the same operations over $\mathbb{R}_\kappa$. Moreover, this notion of computability preserves the correspondence between continuous functions and functions which are computable with an oracle.

A second natural continuation of this paper is the systematic study of the real analysis of $\mathbb{R}_\kappa$. Particularly interesting would be the study of a notion of integral. This problem is not new in the theory of surreal numbers and partial solutions have been proposed in the last decades (see [11, pp. 2-3]). Recently a solution to the problem of integration over the surreal numbers has been proposed in [11]. We are currently working on the problem of integration over $\mathbb{R}_\kappa$.

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Appendix

In this section we will list the most significant proofs of this paper.

Proofs of section 3.1

Proof (Theorem 9). We have to prove that the three properties of κ-topologies hold for τ_κ:

Note that, $\emptyset, X \in \tau_\kappa$ by definition.

Now we have to show that for every ordinal $\beta < \kappa$ and sequence $(\beta'_\alpha)_{\alpha\in\beta}$ such that

$$\forall\alpha < \beta.\ \beta'_\alpha < \kappa \text{ and } B_\alpha \in B,$$

the set $\bigcup_{\alpha\in\beta}\bigcup_{\alpha'\in\beta'_\alpha} B_\alpha$ is a union of less than κ elements of B. We have

$$|\bigcup_{\alpha<\beta}\{\alpha' \in \kappa \mid \alpha' < \beta_\alpha\}| \leq \sum_{\alpha<\beta}|\{\alpha' \in \kappa \mid \alpha' < \beta_\alpha\}|$$

and by regularity of κ

$$\sum_{\alpha<\beta}|\{\alpha' \in \kappa \mid \alpha' < \beta_\kappa\}| \leq \max\{|\beta|, \sup_{\alpha<\beta}\{|\beta_\alpha|\}\} < \kappa.$$

Then $\bigcup_{\alpha\in\beta}\bigcup_{\alpha'\in\beta'_\alpha} B_{\alpha'}$ is a union of less than κ elements of B as desired.

Finally, let $A, B \in \tau_\kappa$. Then, there are $\alpha, \beta < \kappa$ and two sequences $(A_\gamma)_{\gamma\in\alpha}$ and $(B_\gamma)_{\gamma\in\beta}$ of elements of B such that $A = \bigcup_{\gamma<\alpha} A_\gamma$ and $B = \bigcup_{\gamma<\beta} B_\gamma$. Then we have

$$A \cap B = \bigcup_{\gamma<\alpha} A_\gamma \cap \bigcup_{\gamma<\beta} B_\gamma$$

and therefore

$$A \cap B = \bigcup_{(\gamma,\gamma')\in\alpha\times\beta} (A_\gamma \cap B_{\gamma'}).$$

Now for all $(\gamma, \gamma') \in \alpha \times \beta$ the set $A_\gamma \cap B_{\gamma'}$ is either $\emptyset$ or in B and since $\alpha, \beta < \kappa$, we have $A \cap B \in \tau_\kappa$ as desired.

Proof (Lemma 16). Assume $f(X)$ not κ-connected. Therefore there are U, V κ-open subsets of Y which partition $f(X)$. By the κ-continuity of f we have that $f^{-1}(U)$ and $f^{-1}(V)$ are κ-open subsets of X. Moreover, since $f(X) = U \cup V$ we have that $f^{-1}(U)$ and $f^{-1}(V)$ separates X, but this contradicts our hypothesis therefore $f(X)$ is κ-connected.

Proofs of section 3.2

Proof (IVT^K_κ). We can assume $f(a) \neq r \neq f(b)$. Assume that there is no $c \in [a,b]$ such that $f(c) = r$. We define two sets

$$A = f([a,b]) \cap (-\infty, r) \text{ and } B = f([a,b]) \cap (r, +\infty).$$

They are non empty and disjoint ($f(a) \in A$ and $f(b) \in B$). By definition they are also κ-open sets in $f([a,b])$. Moreover, since there is no c such that $f(c) = r$, we have that $f([a,b]) = A \cup B$. Hence A and B separates $f([a,b])$. Now $[a,b]$ is κ-connected and by Theorem 16, $f([a,b])$ is κ-connected. Therefore we have a contradiction since we have shown that A and B separate $f([a,b])$ and $f([a,b])$ is κ-connected.

Proof (Theorem 18). Let p be a polynomial in K and (a,b) be an interval with endpoints in $K \cup \{+\infty, -\infty\}$. Note that since constant functions are κ-continuous we can assume that p is not the zero polynomial. Since K is a real closed field, the polynomials $p(x) - a$ and $p(x) - b$ have finitely many (possibly 0) roots in K. Let $(r_i)_{i \in n}$ be the strictly increasing listing of these roots. Define the set I as follows:

1. if $n = 0$:
$$I = \begin{cases} \{(-\infty, +\infty)\} & \text{If there is } x \in K \text{ s.t. } p(x) \in (a,b), \\ \{\emptyset\} & \text{otherwise.} \end{cases}.$$

2. If $n > 0$: define I as follows:
   (a) $(r_i, r_{i+1}) \in I$ iff $p(\frac{r_{i+1} - r_i}{2}) \in (a,b)$.
   (b) $(-\infty, r_0) \in I$ iff $p(r_0 - 1) \in (a,b)$.
   (c) $(r_{n-1}, +\infty) \in I$ iff $p(r_0 + 1) \in (a,b)$.

Now we claim that $p^{-1}[(a,b)] = \bigcup I$. We will first prove that $p^{-1}[(a,b)] \subseteq \bigcup I$.

Let $x \in p^{-1}[(a,b)]$. If $n = 0$ then trivially $x \in (-\infty, +\infty) = \bigcup I$. Assume $n > 0$. We have the following cases:

Assume that there is $i < n$ such that $r_i < x < r_{i+1}$, we want to prove $p(\frac{r_{i+1} - r_i}{2}) \in (a,b)$. Assume not. Since, K is a real closed field, by [17, Theorem 3.3.9] , the IVT holds for polynomials. In particular, since $p(\frac{r_{i+1} - r_i}{2}) \notin (a,b)$ and $p(x) \in (a,b)$ either $p(x) - a$ or $p(x) - b$ has a root in between r_i and r_{i+1}. But this is in contradiction whit the fact that $(r_i)_{i \in n}$ was strictly increasing. Therefore $p(\frac{r_{i+1} - r_i}{2}) \in (a,b)$ and $(r_i, r_{i+1}) \subseteq \bigcup I$.

Assume that for every $i < n$, we have $x < r_i$. We want to prove $p(r_0 - 1) \in (a,b)$. Assume not. As before the IVT holds for polynomials in K. In particular, since $p(r_0 - 1) \notin (a,b)$ and $p(x) \in (a,b)$ either $p(x) - a$ or $p(x) - b$ has a root in between $-\infty$ and r_0. But this is in contradiction whit the fact that $(r_i)_{i \in n}$ was the strictly increasing listing of all the roots of $p(x) - a$ and $p(x) - b$. Therefore $p(r_0 - 1) \in (a,b)$ and $(-\infty, r_0) \subseteq \bigcup I$.

Finally if for every $i < n$ we have $x > r_i$, then the proof is similar to the previous case.

Note that the case in which $x = r_i$ for some r_i is impossible since $p(x) \in (a, b)$.

Now we will prove that $p^{-1}[(a, b)] \supseteq \bigcup I$. Let $x \in \bigcup I$. If $n = 0$, then there is $y \in K$ such that $p(y) \in (a, b)$. Now, since the IVT hold for polynomials, if $p(x) \notin (a, b)$ we would have that either $p(x) - a$ or $p(x) - b$ has a root. This contradicts the assumptions. Assume $n > 0$. We will only consider the case in which $x \in (r_i, r_{i+1})$ for some $i < n$ and $p(\frac{r_{i+1}-r_i}{2}) \in (a, b)$. The other cases can be proved similarly. We want to prove that $p(x) \in (a, b)$. Assume not. Since the IVT hold for polynomials, we would have that either $p(x) - a$ or $p(x) - b$ has a root in between (r_i, r_{i+1}). But we assumed $(r_i)_{i\in n}$ strictly increasing. Therefore $p(x) \in (a, b)$ as desired. Therefore $p^{-1}[(a, b)] = \bigcup I$.

Now since I is a finite list of intervals with end points in $K \cup \{-\infty, +\infty\}$ we have that $\bigcup I$ is κ-open. Hence $p^{-1}[(a, b)]$ is κ-open and p is κ-continuous as desired.

Proofs of section 4

Theorem 25 (Gonshor). *Let L and R be two sets of surreal numbers such that $L < R$. Then $\ell([L|R])$ is smaller or equal to the least ordinal α such that $\forall x \in L \cup R.\ \ell(x) < \alpha$.*

Proof. Note that this follows trivially from the fact that $[L|R]$ is defined to be de shortest surreal number strictly between L and R, then if it is of length bigger than α. Hence $[L|R]\restriction\alpha$ would be shorter than $[L|R]$ and still in between L and R.

Proposition 26 (Folklore). *Let κ' be a cardinal such that $\mathrm{Cof}(\kappa') = \alpha$. Then $\mathrm{No}_{<\kappa'}$ is a η_α-set.*

Proof. Assume $L, R \in \mathrm{No}_{<\kappa'}$ such that $|L| + |R| < \kappa'$. Then for every $x \in L$ and $y \in R$ we have $|x|, |y| < \kappa'$. But since $\mathrm{Cof}(\kappa') = \alpha$, we have that

$$\mathrm{Length}(L) = \sup\{\ell(x) \mid x \in L\} \text{ and } \mathrm{Length}(R) = \sup\{\ell(x) \mid x \in R\}$$

are both smaller than κ'. But then by Theorem 25 we have

$$\ell([L|R]) \leq \max\{\mathrm{Length}(L), \mathrm{Length}(R)\} < \kappa'$$

which implies $[L|R] \in \mathrm{No}_{<\kappa'}$ as desired.

Proof (Proposition 19). We want to prove $|\mathrm{No}_{<\kappa}| = \kappa$. Since we assumed $\kappa = \kappa^{<\kappa}$, the statement follows from the fact that $\mathrm{No}_{<\kappa}$ is the set of sequences of pluses and minuses of length less than κ.

To prove $\mathrm{Cof}(\mathrm{No}_{<\kappa}) = \mathrm{Coi}(\mathrm{No}_{<\kappa}) = \kappa$, note that $\kappa \subset \mathrm{No}_{<\kappa}$ is a cofinal subset of $\mathrm{No}_{<\kappa}$ and $-\kappa$ is a coinitial subset of $\mathrm{No}_{<\kappa}$. By regularity of κ, $\mathrm{Cof}(\mathrm{No}_{<\kappa}) = \mathrm{Coi}(\mathrm{No}_{<\kappa}) = \kappa$. Moreover, note that every dense subset of $\mathrm{No}_{<\kappa}$ has to be cofinal in $\mathrm{No}_{<\kappa}$, therefore the weight of $\mathrm{No}_{<\kappa}$ is less than or equal to κ and since $|\mathrm{No}_{<\kappa}| = \kappa$, we have that the weight of $(\mathrm{No}_{<\kappa})$ is κ.

We know that $\mathrm{No}_{<\kappa}$ is a real closed field. Consider the following sequence $S = \{\frac{1}{\alpha}\}_{\alpha\in\kappa}$. The sequence S is coinitial in $\mathrm{No}^+_{<\kappa}$. Indeed, take $x \in \mathrm{No}^+_{<\kappa}$. We can assume $x < 1$, therefore $x = \frac{1}{y}$ with $y \in \mathrm{No}^+_{<\kappa}$. Take $\alpha < \kappa$ such that $\alpha > y$ (note that α exists since κ is cofinal in $\mathrm{No}_{<\kappa}$), then $x > \frac{1}{\alpha} > 0$ and S is coinitial in $\mathrm{No}^+_{<\kappa}$. Now, since $|S| = \kappa$, therefore $\mathrm{Coi}(\mathrm{No}^+_{<\kappa}) \leq \kappa$. Moreover, note that any subsequence S' of $\mathrm{No}^+_{<\kappa}$ of cardinality less than κ cannot be coinitial in $\mathrm{No}^+_{<\kappa}$. Indeed, let S' be such a sequence. Since $\mathrm{No}_{<\kappa}$ is an η_κ-set, if we take $L = \{0\}$ and $R = S'$, there is $x \in \mathrm{No}_{<\kappa}$ such that $L < \{x\} < R$. Trivially $x \in \mathrm{No}^+_{<\kappa}$ and $\{x\} < S'$. Hence S' is not coinitial in $\mathrm{No}^+_{<\kappa}$ as desired. In conclusion $\mathrm{Deg}(\mathrm{No}^+_{<\kappa}) = \mathrm{Coi}(\mathrm{No}^+_{<\kappa}) = \kappa$.

Lemma 27. *Let $\langle L, R\rangle$ be a Veronese cut over $\mathbb{R}_\kappa$. There are two sequences $(\ell_\gamma)_{\gamma\in\alpha}$ and $(r_\gamma)_{\gamma\in\beta}$ of elements of $\mathrm{No}_{<\kappa}$ with*

$$\alpha \leq \min\{|L|, \kappa\} \textit{ and } \beta \leq \min\{|R|, \kappa\}$$

such that

$$\langle \bigcup_{\gamma\in\alpha}\{\ell_\gamma\} \mid \bigcup_{\gamma\in\beta}\{r_\gamma\}\rangle$$

is a Veronese cut and

$$[\bigcup_{\gamma\in\alpha}\{\ell_\gamma\} \mid \bigcup_{\gamma\in\beta}\{r_\gamma\}] = [L|R].$$

Proof. Let $\langle L, R\rangle$ be a Veronese cut. We claim that there are two sequences $(\ell_\gamma)_{\gamma\in\alpha}$ and $(r_\gamma)_{\gamma\in\beta}$ of elements of $\mathrm{No}_{<\kappa}$ with

$$\alpha \leq \min\{|L|, \kappa\} \text{ and } \beta \leq \min\{|R|, \kappa\},$$

which are respectively mutually cofinal in L and mutually coinitial in R. Moreover, since L has no maximum and R has no minimum, we can choose $(\ell_\gamma)_{\gamma\in\alpha}$ and $(r_\gamma)_{\gamma\in\beta}$ such that $\ell_\gamma < r_{\gamma'}$ for all $\gamma \in \alpha$ and $\gamma' \in \beta$. Let $\ell \in L$ and $r \in R$ be two elements respectively of L and R. Since L has no maximum and R has no minimum therefore there exist $\ell' \in L$ and $r' \in R$ such that $\ell < \ell'$ and $r' < r$. By the density of $\mathrm{No}_{<\kappa}$ in $\mathbb{R}_\kappa$ there are $\ell_0, r_0 \in \mathrm{No}_{<\kappa}$ such that $\ell < \ell_0 < \ell' < r' < r_0 < r$. Now let $0 < \gamma < \kappa$ and assume we have already defined $\ell_{\gamma'}$, for every $\gamma' < \gamma$. We will define ℓ_γ, the same argument works for r_γ. We have two cases:

1. if there exists $\ell \in L$ such that $\ell_{\gamma'} < \ell$ for all $\gamma' < \gamma$, then take $\ell' \in L$ such that $\ell < \ell'$ and $\ell_\gamma \in \mathrm{No}_{<\kappa}$ such that $\ell < \ell_\gamma < \ell'$.
2. If for all $\ell \in L$ there is $\gamma' < \gamma$ such that $\ell_{\gamma'} \geq \ell$ stop.

Now let α be the smallest ordinal on which the previous definition stops. Note that trivially $\alpha \leq \min\{|L|, \kappa\}$. It is an easy induction to prove that for every $\gamma < \alpha$ there are $\ell, \ell' \in L$ such that $\ell < \ell_\gamma < \ell'$ and that for every $\ell \in L$ there is $\gamma \in \alpha$ such that $\ell \leq \ell_\gamma$. Therefore $(\ell_{\gamma'})_{\gamma'<\alpha}$ is mutually cofinal with L as desired.

Now by a standard cofinality argument we have

$$[\bigcup_{\gamma\in\alpha}\{\ell_\gamma\} \mid \bigcup_{\gamma\in\beta}\{r_\gamma\}] = [L|R].$$

Moreover, since $\langle L, R\rangle$ is a Veronese cut, $\langle\bigcup_{\gamma\in\alpha}\{\ell_\gamma\}, \bigcup_{\gamma\in\beta}\{r_\gamma\}\rangle$ is a Veronese representation in $\mathrm{No}_{<\kappa}$. Finally, since $[L|R] \in \mathbb{R}_\kappa$, we have that

$$[\bigcup_{\gamma\in\alpha}\{\ell_\gamma\} \mid \bigcup_{\gamma\in\beta}\{r_\gamma\}]$$

is in $\mathbb{R}_\kappa$ as desired.

Proof (Lemma 22). Let $x, y \in \mathbb{R}_\kappa$ be such that $x < y$. We can assume that at least one between x and y is not in $\mathrm{No}_{<\kappa}$, otherwise the statement follows trivially by the density of $\mathrm{No}_{<\kappa}$. Without loss of generality assume y is not in $\mathrm{No}_{<\kappa}$. Let $[L_x|R_x]$ be the standard representation of x and $[L_y|R_y]$ be a representation of y such that $\langle L_y, R_y\rangle$ is Cauchy. Since $x < y$, by [14, Theorem 2.5] we have $\{x\} < R_y$ and $\{y\} > L_x$. Moreover, since $x \neq y$, by [14, Theorem 2.6] we have that either there exists $x_R \in R_x$ such that $y \geq x_R$ or exists $y_L \in L_x$ such that $y_L \geq x$. Assume that there is $x_R \in R_x$ such that $y \geq x_R$. Since $y_L \notin \mathrm{No}_{<\kappa}$ and $y \neq x_R$, we have $y > x_R > x$ as desired. On the other hand if there exists $y_L \in L_x$ such that $y_L \geq x$, then by the fact that L_y has no maximum we can take $y > y'_L > y_L \geq x$. Therefore y'_L is the desired element of $\mathrm{No}_{<\kappa}$.

The fact that $\mathbb{R}_\kappa$ is Cauchy complete follows trivially by Lemma 27.

Lemma 28. *Let* $\mathrm{No}_{\leq\kappa}$ *be the set of surreal numbers of length at most* κ*. Then* $\mathbb{R}_\kappa \subseteq \mathrm{No}_{\leq\kappa}$.

Proof. We will prove that $\mathrm{No}_{\leq\kappa}$ contains the Dedekind closure of $\mathbb{Q}_\kappa$. This implies by definition that $\mathrm{No}_{\leq\kappa}$ also contains the Cauchy closure of $\mathbb{Q}_\kappa$, namely $\mathbb{R}_\kappa$. Let $\langle L, R\rangle$ be a cut in $\mathbb{Q}_\kappa$, we claim that $[L|R] \in \mathrm{No}_{\leq\kappa}$. Note that for every $x \in L \cup R$, since $L \cup R \subset \mathbb{Q}_\kappa$, we have $\ell(x) < \kappa$. Therefore, by Theorem 25 we have $\ell([L|R]) \leq \kappa$. Then $[L|R] \in \mathrm{No}_{\leq\kappa}$ as desired. Now, since $\mathbb{R}_\kappa = \mathbb{Q}_\kappa \cup V$ where V is the set of Veronese cuts over $\mathbb{Q}_\kappa$, we have $\mathbb{R}_\kappa \subseteq \mathrm{No}_{\leq\kappa}$ as desired.

Proof (Theorem 24).

We will begin proving that $\mathrm{Deg}(\mathbb{Q}_\kappa) = \mathrm{Deg}(\mathbb{R}_\kappa)$. Since $\mathbb{Q}_\kappa$ is dense in $\mathbb{R}_\kappa$, then we have that $\mathrm{Deg}(\mathbb{R}_\kappa) \geq \mathrm{Deg}(\mathbb{Q}_\kappa)$. Now, assume that every sequence of length κ in $\mathbb{R}_\kappa$ is such that there is $x \in \mathbb{R}_\kappa^+$ smaller than every element of the sequence. Then, by the density of $\mathbb{Q}_\kappa$, there is $x' \in \mathbb{Q}_\kappa$ such that $0 < x' < x$, but this is absurd because $\mathrm{Deg}(\mathbb{Q}_\kappa) = \kappa$.

Now we want to prove that $\mathbb{R}_\kappa$ is an η_κ-set. Take $L, R \subset \mathbb{R}_\kappa$ such that $|L| + |R| < \kappa$ and $L < R$. We have the following possibilities:

- L has no maximum and R has no minimum: by the density of $\mathbb{Q}_\kappa$ in $\mathbb{R}_\kappa$, by Lemma 27, there are two sequences $(\ell_\gamma)_{\gamma\in\alpha}$ and $(r_\gamma)_{\gamma\in\beta}$ with $\alpha, \beta < \kappa$

of elements of $\mathbb{Q}_\kappa$ which are respectively cofinal in L and and coinitial in R and such that

$$\forall \alpha' \in \alpha \forall \beta' \in \beta .\ \ell_{\alpha'} < r_{\beta'}.$$

Hence, since $\mathbb{Q}_\kappa$ is an η_κ-set, we have that there is $x \in \mathbb{R}_\kappa$ such that

$$L < \{x\} < R$$

as desired.

- L has maximum M and R has minimum m: it is enough to take $x = \frac{m-M}{2}$.
- L has maximum M and R has no minimum: consider the sequence

$$(r - M)_{r \in R}.$$

Note that

$$\forall r \in R.\ r - M > 0,$$

therefore, since $\mathrm{Deg}(\mathbb{R}_\kappa) = \kappa$ and $|R| < \kappa$ there is $x \in \mathbb{R}_\kappa$ such that

$$\forall r \in R.\ 0 < x < r - M,$$

but then $M < x + M$ and $\forall r \in R.\ x + M < r$ as desired.
- L has no maximum M and R has minimum m: a proof similar to the previous case applies.

Note that by the construction of $\mathbb{R}_\kappa$, since $\mathbb{Q}_\kappa$ is a dense subfield of $\mathbb{R}_\kappa$, we have

$$\mathrm{Cof}(\mathbb{R}_\kappa) = \mathrm{Coi}(\mathbb{R}_\kappa) = \kappa \text{ and } \mathbb{R}_\kappa \text{ has weight } \kappa.$$

Now, we want to prove $2^\kappa \leq |\mathbb{R}_\kappa| \leq 2^\kappa$. On the one hand, by Lemma 28 we have that $\mathbb{R}_\kappa \subset \mathrm{No}_{\leq\kappa}$. Indeed, $\mathrm{No}_{\leq\kappa}$ contains the Dedekind completion of $\mathbb{Q}_\kappa$, hence also its Cauchy completion $\mathbb{R}_\kappa$. Then, since $|\mathrm{No}_{\leq\kappa}| = 2^\kappa$, we have that $|\mathbb{R}_\kappa| \leq 2^\kappa$.

On the other hand let $\{0,1\}^{<\kappa}$ be the full binary tree of height κ, we define a tree T which is in bijection with $\{0,1\}^{<\kappa}$ and whose nodes are subsets of $\mathbb{R}_\kappa$ and whose branches corresponds to different elements of $\mathbb{R}_\kappa$. We define the tree by recursion as follows:

$$\text{Set } T_\lambda = (0,1) \text{ as the root of the tree.}$$

Now assume that for $p \in 2^{<\kappa}$, the element $T_p \neq \emptyset$ is an already defined open interval in $\mathbb{R}_\kappa$. We define T_{p0} and T_{p1} such that:

$$\begin{aligned}
&T_{p0} \cup T_{p1} \subseteq T_p,\\
&T_{p0} \neq \emptyset \text{ and } T_{p1} \neq \emptyset,\\
&T_{p0} \cap T_{p1} = \emptyset,\\
&T_{p0} = (a_{p0}, b_{p0}) \text{ and } T_{p1} = (a_{p1}, b_{p1}),
\end{aligned}$$

with $a_{p0}, b_{p0}, a_{p1}, b_{p1} \in \mathbb{Q}_\kappa$ such that

$$|a_{p0} - b_{p0}| \leq \frac{1}{\ell(p)+1} \text{ and } |a_{p1} - b_{p1}| \leq \frac{1}{\ell(p)+1}.$$

Finally if $p \in 2^{<\kappa}$ is of limit length γ and $T_{p \restriction \alpha}$ has already been defined for every $\alpha < \gamma$, we define T'_p as follows:

$$T'_p = \bigcap_{\alpha<\gamma} T_{p \restriction \alpha}.$$

Note that by the fact that $\mathbb{R}_\kappa$ is an η_κ-set, the set T_p is never empty, moreover T'_p is trivially convex (i.e. if $x, y \in T'_p$ and $x \leq z \leq y$, then $z \in T'_p$). Therefore we can define T_p as follows:

$$\begin{aligned}
&T_p \subseteq T'_p, \\
&T_p \neq \emptyset, \\
&T_p = (a_p, b_p) \text{ with } a_p, b_p \in \mathbb{Q}_\kappa \text{ such that } |a_p - b_p| \leq \frac{1}{\ell(p)}.
\end{aligned}$$

It follows trivially by the way in which we have defined the tree that the set $\bigcap_{\alpha\in\kappa} T_{p\restriction\alpha}$ contains a single element of $\mathbb{R}_\kappa$. Indeed by the properties of the tree we have that $[\{a_{p\restriction\alpha} \mid \alpha \in \kappa\} \mid \{b_{p\restriction\alpha} \mid \alpha \in \kappa\}]$ is a Veronese cut in $\mathbb{Q}_\kappa$. Therefore $\bigcap_{\alpha\in\kappa} T_{p\restriction\alpha}$ is a singleton in $\mathbb{R}_\kappa$ as desired. Therefore we have $2^\kappa \leq |\mathbb{R}_\kappa|$ as desired.