BEST RATIONAL APPROXIMATION OF FUNCTIONS WITH LOGARITHMIC SINGULARITIES

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Abstract. We consider functions ω on the unit circle $\mathbb{T}$ with a finite number of logarithmic singularities. We study the approximation of ω by rational functions and find an asymptotic formula for the distance in the BMO-norm between ω and the set of rational functions of degree n as $n \to \infty$. Our approach relies on the Adamyan-Arov-Krein theorem and on the study of the asymptotic behaviour of singular values of Hankel operators.

1. Introduction

1.1. **Overview.** The rate of convergence of both rational and polynomial approximations to a given function φ is determined by the smoothness of φ. Of course in general the rational approximations converge much faster than the polynomial ones.

Let us briefly describe the fundamental results of approximation theory relevant to this paper; see [22] for more information. We denote by $\mathcal{P}_n$ the set of all polynomials in $x \in \mathbb{R}$ of degree $\leq n$. Similarly, $\mathcal{T}_n$ is the set of all trigonometric polynomials of degree $\leq n$ defined on the unit circle $\mathbb{T}$. According to the classical Jackson-Bernstein theorem (see, e.g., the book [5]), the distance between φ and $\mathcal{T}_n$ in the L^∞-norm satisfies the estimate

$$\operatorname{dist}_{L^\infty(\mathbb{T})}\{\varphi, \mathcal{T}_n\} = O(n^{-\alpha}), \quad \alpha > 0, \quad n \to \infty,$$

if and only if φ belongs to the Hölder-Zygmund class Λ_α (the definitions of relevant function classes are collected in the Appendix).

Further, for the function $\varphi(x) = |x|^\alpha$ defined on some interval of the real line, for example on $[-1, 1]$, S. N. Bernstein [2, 3] proved the existence of the limit

$$\lim_{n\to\infty} n^\alpha \operatorname{dist}_{L^\infty(-1,1)}\{|x|^\alpha, \mathcal{P}_n\} = \mathsf{b}(\alpha) \tag{1.1}$$

where $\mathsf{b}(\alpha) \neq 0$ if $\alpha \neq 2, 4, \ldots$. The number $\mathsf{b}(\alpha)$ is known as the *Bernstein constant.*

Next, consider the problem of rational approximation. The degree of a rational function p/q (p, q are polynomials with no non-constant common divisors) is defined as $\max\{\deg p, \deg q\}$. We denote by $\mathcal{R}_n$ the set of all rational functions of

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degree $\leq n$ in the complex plane. D. Newman proved in [14] that for the function $\varphi(x) = |x|$ on the interval $[-1,1]$, the distance between φ and $\mathcal{R}_n$ in the L^∞ norm satisfies the estimates

$$e^{-c_1\sqrt{n}} \leq \operatorname{dist}_{L^\infty(-1,1)}\{\varphi, \mathcal{R}_n\} \leq e^{-c_2\sqrt{n}}$$

with some positive constants c_1, c_2. This result was extended by A. A. Gonchar in [8, 9] to the functions $\varphi(x) = |x|^\alpha$; he established the same estimate for all $\alpha > 0$, $\alpha \neq 2, 4, \ldots$. More recently, H. Stahl [22] proved a remarkable result: he showed that for such functions one has the asymptotic relation

$$\lim_{n\to\infty} e^{\pi\sqrt{\alpha n}} \operatorname{dist}_{L^\infty(-1,1)}\{|x|^\alpha, \mathcal{R}_n\} = 4^{1+\alpha/2}|\sin \tfrac{1}{2}\pi\alpha|, \quad \alpha > 0; \tag{1.2}$$

see [22] for the history of the problem.

In this paper, we discuss the rational approximation of functions with logarithmic singularities of the type $(-\ln|x|)^{-\alpha}$ near $x = 0$. More precisely, let us fix $\alpha > 0$ and consider the function

$$\varphi_+(x) = \begin{cases} (-\ln x)^{-\alpha} & x \in (0, 1/2] \\ 0 & x \in [-1/2, 0] \end{cases}$$

on the interval $[-1/2, 1/2]$. Clearly, φ_+ does not satisfy the Hölder continuity condition (with any exponent) and so, according to the Jackson-Bernstein theorem, $\operatorname{dist}_{L^\infty}\{\varphi_+, \mathcal{P}_n\}$ goes to zero slower than any power of n^{-1}. On the other hand, A. A. Gonchar in [9] proved the two-sided estimates

$$cn^{-\alpha} \leq \operatorname{dist}_{L^\infty(-1/2,1/2)}\{\varphi_+, \mathcal{R}_n\} \leq C(\ln n/n)^\alpha \tag{1.3}$$

with some positive constants c and C.

Our aim is to obtain an asymptotic relation for the function $\varphi_+(x)$ (and for more general functions with similar singularities) in the spirit of (1.2) but with n^α instead of the exponential $e^{\pi\sqrt{\alpha n}}$. We obtain such a relation, but for the BMO-norm instead of the L^∞-norm. In fact, in harmonic analysis the space BMO (functions with bounded mean oscillation) often plays the role of a proper substitute for L^∞; this space is only slightly larger than L^∞ and $L^\infty \subset \mathrm{BMO} \subset L^p$ for any $p < \infty$. Further, this space is particularly well adapted to treating functions with logarithmic singularities and allows us to study unbounded functions. Indeed, along with φ_+ we consider the function

$$\varphi_0(x) = (-\ln|x|)^{1-\alpha}, \quad x \in [-1/2, 1/2],$$

which *a priori* looks more singular than φ_+ because of the extra factor $\ln|x|$. For $0 < \alpha < 1$, this function is unbounded, but it is in the VMO class (functions with vanishing mean oscillation). Just as for φ_+, we show that $n^\alpha \operatorname{dist}_{\mathrm{BMO}}\{\varphi_0, \mathcal{R}_n\}$ attains a finite positive limit as $n \to \infty$ which we compute explicitly. Comparing these facts with the classical Bernstein result (1.1), we see that, for functions with logarithmic singularities, rational approximations play the same role as polynomial approximations play for functions with power singularities.

It will be convenient for us to work with functions defined on the unit circle $\mathbb{T}$ of the complex plane, instead of on an interval of the real line. So, to be more precise, below we consider the analogues of φ_+ and φ_0 on the unit circle.

Our approach relies on a combination of the fundamental Adamyan-Arov-Krein (AAK) theorem [1] and of our previous results [19, 20] on the asymptotic behaviour of singular values of Hankel operators of a certain class. The AAK theorem relates the rational approximation of a function φ (defined on $\mathbb{T}$) in the BMO-norm to the singular values of the Hankel operator with the symbol φ. This explains why we work with the BMO-norm rather than with the L^∞-norm.

Using the AAK theorem, V. V. Peller [16] has obtained the following analogue of the Jackson-Bernstein theorem for the rational approximations in the BMO norm. He proved that

$$\operatorname{dist}_{\mathrm{BMO}(\mathbb{T})}\{\varphi, \mathcal{R}_n\} = O(n^{-\alpha}), \quad \alpha > 0, \quad n \to \infty, \tag{1.4}$$

if and only if φ belongs to a certain Besov-Lorentz class, denoted by $\mathfrak{B}^{\alpha}_{1/\alpha,\infty}$ in [16]. We reproduce the definition of this class in the Appendix. Of course, the specific functions φ with logarithmic singularities that we consider in this paper belong to this class.

1.2. **BMO and VMO.** We denote by $\mathbb{T}$ the unit circle in the complex plane, equipped with the normalized Lebesgue measure $dm(\mu) = (2\pi i\mu)^{-1}d\mu$, $\mu \in \mathbb{T}$, and set $L^p \equiv L^p(\mathbb{T})$. For $f \in L^1$, let

$$\widehat{f}(j) = \int_{\mathbb{T}} f(\mu)\mu^{-j} dm(\mu), \quad j \in \mathbb{Z},$$

be the Fourier coefficients of f. For $1 \le p \le \infty$, the Hardy classes H^p_+ and H^p_- are defined in a standard way as

$$H^p_+ = \{f \in L^p(\mathbb{T}) : \widehat{f}(j) = 0 \quad \forall j < 0\}, \quad H^p_- = \{f \in L^p(\mathbb{T}) : \widehat{f}(j) = 0 \quad \forall j \ge 0\}.$$

We denote by $P_+ : L^2 \to H^2_+$ and $P_- : L^2 \to H^2_-$ the orthogonal projections onto H^2_+ and H^2_-. There is a lack of complete symmetry between H^2_+ and H^2_- because the constant functions belong to H^2_+ but not to H^2_-. This results in a slight asymmetry in some of the formulas below.

The class $\mathrm{BMO}(\mathbb{T}) =: \mathrm{BMO}$ can be described in many equivalent ways, with equivalent choices for the norm; see, e.g., [11]. Since we are interested in the *asymptotics* of the distance in the BMO-norm, the precise choice of the norm will be important for us.

Let us start by fixing the BMO-norm for functions analytic outside the unit disk. A function $f \in H^2_-$ belongs to BMO if and only if $f - g \in L^\infty$ for some $g \in H^2_+$; then we set

$$\|f\|_{\mathrm{BMO}} = \inf\{\|f - g\|_{L^\infty} : g \in H^2_+\}, \quad f \in H^2_-. \tag{1.5}$$

Remark. Let $H^1_+(0) = \{f \in H^1_+ : \widehat{f}(0) = 0\}$. As is well known (see, e.g. [11, Section VII.A.1]), the norm (1.5) coincides with the norm of f in the dual space $H^1_+(0)^*$.

Next, for an arbitrary $f \in L^2$ we set

$$\|f\|_{\mathrm{BMO}} = \max\{\|P_- f\|_{\mathrm{BMO}}, \|P_- \bar{f}\|_{\mathrm{BMO}}, |\widehat{f}(0)|\}, \tag{1.6}$$

if the right hand side is finite. Here $\|P_- f\|_{\mathrm{BMO}}$, $\|P_- \bar{f}\|_{\mathrm{BMO}}$ are defined by (1.5). Clearly, the norm (1.6) is invariant with respect to the complex conjugation,

$$\|\overline{f}\|_{\mathrm{BMO}} = \|f\|_{\mathrm{BMO}}.$$

With this definition, we have

$$\|P_+ f\|_{\mathrm{BMO}} = \max\{\|P_- \bar{f}\|_{\mathrm{BMO}}, |\widehat{f}(0)|\}, \tag{1.7}$$

$$\|f\|_{\mathrm{BMO}} = \max\{\|P_+ f\|_{\mathrm{BMO}}, \|P_- f\|_{\mathrm{BMO}}\}. \tag{1.8}$$

Finally, we recall that the subclass VMO $\subset$ BMO is the closure of all continuous functions in the BMO-norm.

Let us comment on our choice of the norm in BMO. Definition (1.5) is absolutely crucial for our approach, as it ensures the connection of rational approximations with Hankel operators via the AAK theorem. On the other hand, the details of the definition (1.6) are less important: the term $|\widehat{f}(0)|$ is inessential and the other two quantities in the right-hand side can be combined in various ways. Our choice (1.6) is motivated by the fact that it simplifies the expressions for some coefficients appearing in the asymptotic formulas below. We could have chosen, for example, the following alternative definition of the BMO norm:

$$\|f\|^*_{\mathrm{BMO}} = \|P_- f\|_{\mathrm{BMO}} + \|P_- \bar{f}\|_{\mathrm{BMO}} + |\widehat{f}(0)|;$$

this would only change the constant in the right-hand side of some asymptotic formulas, such as (1.15) below.

1.3. **Rational approximation.** We denote by $\mathcal{R}_n$ the set of all rational functions of degree $\leq n$ in the complex plane without poles on $\mathbb{T}$ and set

$$\mathcal{R}^\pm_n = \mathcal{R}_n \cap H^2_\pm = P_\pm(\mathcal{R}_n).$$

Notice that $\mathcal{R}^+_0 = \{\mathrm{const}\}$, while $\mathcal{R}^-_0 = \{0\}$.

For $\omega \in$ BMO and $n \geq 0$, we define

$$\begin{aligned} \rho_n(\omega) &= \mathrm{dist}_{\mathrm{BMO}}\{\omega, \mathcal{R}_n\} := \min\{\|\omega - r\|_{\mathrm{BMO}} : r \in \mathcal{R}_n\}, \\ \rho^+_n(\omega) &= \mathrm{dist}_{\mathrm{BMO}}\{P_+\omega, \mathcal{R}_n\} = \mathrm{dist}_{\mathrm{BMO}}\{P_+\omega, \mathcal{R}^+_n\}, \\ \rho^-_n(\omega) &= \mathrm{dist}_{\mathrm{BMO}}\{P_-\omega, \mathcal{R}_n\} = \mathrm{dist}_{\mathrm{BMO}}\{P_-\omega, \mathcal{R}^-_n\}. \end{aligned} \tag{1.9}$$

There are some simple identities relating the quantities ρ_n, ρ^+_n, ρ^-_n, see Lemma 2.7 below. It is clear that $\omega \in$ VMO (resp. $P_\pm\omega \in$ VMO) if and only if $\rho_n(\omega) \to 0$ (resp. $\rho^\pm_n(\omega) \to 0$) as $n \to \infty$.

From our choice of the BMO-norm it follows that $\rho_n^-(\omega)$ can be alternatively written as

$$\rho_n^-(\omega) = \operatorname{dist}_{L^\infty}\{\omega, \mathcal{R}_n^- + H_+^2\}. \tag{1.10}$$

The problem of approximation by functions of the class $\mathcal{R}_n^- + H_+^2$ in L^∞-norm is known as the Nehari-Takagi problem. Of course, a similar statement is true for $\rho_n^+(\omega)$ (see Lemma 2.1 below):

$$\rho_n^+(\omega) = \operatorname{dist}_{L^\infty}\{\omega, \mathcal{R}_n^+ + H_-^2\}. \tag{1.11}$$

1.4. **Outline of results.** To give the flavour of our main result, first we consider the following model functions:

$$\omega_0(e^{i\theta}) = |\log|\theta||^{1-\alpha}\chi_0(\theta), \qquad \theta \in [-\pi, \pi), \tag{1.12}$$

$$\omega_\pm(e^{i\theta}) = |\log|\theta||^{-\alpha}\chi_0(\theta)\mathbb{1}_\pm(\theta), \qquad \theta \in [-\pi, \pi). \tag{1.13}$$

Here $\alpha > 0$ is a fixed parameter, $\mathbb{1}_\pm$ is the characteristic function of the semi-axis $\mathbb{R}_\pm$, and χ_0 is a smooth even cutoff function, whose role is to remove the "undesired" singularity of the functions (1.12) and (1.13) at $|\theta| = 1$. More precisely, $\chi_0 \in C_0^\infty(\mathbb{R})$ is a function which vanishes identically for $|\theta| > c$ with some $c < 1$ and such that $\chi_0(\theta) = 1$ in some neighborhood of zero.

The following statement is a particular case of Theorem 3.5 below. We set

$$\varkappa(\alpha) = 2^{-\alpha}\pi^{1-2\alpha}B(\tfrac{1}{2\alpha}, \tfrac{1}{2})^\alpha \tag{1.14}$$

where $B(\cdot, \cdot)$ is the Beta function.

Theorem 1.1. *Let $\alpha > 0$ and let*

$$\omega(\mu) = v_0\omega_0(\mu) + v_+\omega_+(\mu) + v_-\omega_-(\mu),$$

where v_0, v_+, v_- are arbitrary complex numbers. Put

$$b^\pm = \tfrac{1}{2}(1-\alpha)v_0 \pm \tfrac{1}{2\pi i}(v_+ - v_-).$$

Then

$$\lim_{n\to\infty} n^\alpha \rho_n^\pm(\omega) = \varkappa(\alpha)|b^\pm|,$$

$$\lim_{n\to\infty} n^\alpha \rho_n(\omega) = \varkappa(\alpha)\big(|b^+|^{1/\alpha} + |b^-|^{1/\alpha}\big)^\alpha. \tag{1.15}$$

Remark 1.2. 1. Similarly to Stahl's formula (1.2), the asymptotic coefficient $\varkappa(\alpha)$ in Theorem 1.1 is quite explicit. In contrast to this, it is not known whether the Bernstein constant $\mathsf{b}(\alpha)$ in (1.1) can be expressed in terms of standard transcendentals; see, e.g. [12] for more on this issue.
2. For $0 < \alpha < 1$, the function ω_0 is not in L^∞, although $\omega_0 \in$ VMO.
3. For $\alpha = 1$, the function $\omega_0 = \chi_0 \in C^\infty$. This agrees with the fact that in this case ω_0 makes no contribution to $b^\pm$.
4. For $\alpha = 0$, the functions ω_0 and $\omega_\pm$ are in BMO but not in VMO. Thus, in this case $\rho_n(\omega)$ (and $\rho_n^\pm(\omega)$) do not tend to zero as $n \to \infty$. This shows that one cannot go beyond $\alpha > 0$ in Theorem 1.1.

5. Consider the case $v_0 = 0$; then the asymptotic coefficients $b^\pm$ vanish precisely when $v_+ = v_-$, i.e. when $\omega(e^{i\theta})$ is an even function of θ. This partially explains the fact that although the even function ω_0 is more singular than ω_+ and ω_-, the rational approximations of these functions have the same power rate of convergence.
6. Comparing Theorem 1.1 with Peller's result (1.4), we see that the functions ω_0 and $\omega_\pm$ belong to the Besov-Lorentz class $\mathfrak{B}^\alpha_{1/\alpha,\infty}$, but do not belong to the class $\mathfrak{B}^\beta_{1/\beta,\infty}$ for any $\beta > \alpha$. This yields explicit (and sharp!) examples of functions in these Besov-Lorentz classes.
7. The analogue of Theorem 1.1 for L^∞-distances remains an open problem.

For functions $\omega \in \mathrm{BMO} \cap H^2_\pm$, the distances $\rho^\pm_n(\omega) = \rho_n(\omega)$ do not depend on the choice of the norm (1.6) — see relations (1.10) and (1.11). Therefore for functions analytic for $|z| < 1$ (or for $|z| > 1$), Theorem 1.1 is stated in a quite intrinsic form. Consider, for example, the function

$$\omega(z) = (-\log(1-z) + c)^{1-\alpha}, \quad \alpha > 0, \tag{1.16}$$

where a number c is chosen in such a way that $\log(1-z) \neq c$ for all z with $|z| \leq 1$. Then $\omega(z)$ is analytic in the unit disk $\mathbb{D}$ and is singular only at the point $z = 1$ on the unit circle.

The following statement is a particular case of Theorem 3.8 below.

Theorem 1.3. *Let the function $\omega(z)$ be defined by formula (1.16). Then there exists*

$$\lim_{n\to\infty} n^\alpha \rho^+_n(\omega) = |1-\alpha|\varkappa(\alpha).$$

Two sided estimates by $c\,n^{-\alpha}$ of $\rho^+_n(\omega)$ for the function (1.16) (and for more general functions of this type) are known. They were obtained by A. A. Pekarskiĭ in [17] (see Example 2.2). Later (see [18], relation (31)) Pekarskiĭ also proved the upper bound in the L^∞-norm, $\mathrm{dist}_{L^\infty}\{\omega, \mathcal{R}^+_n\} = O(n^{-\alpha})$.

We emphasize that our main results, Theorems 3.3 and 3.8 below, allow for an arbitrary finite number of logarithmic singularities of ω on the unit circle.

1.5. **Some ideas of the approach.** We start by recalling some concepts related to Hankel operators; for the details, see, e.g., the books [15, 16]. For $\omega \in L^2$, the Hankel operator $K(\omega) : H^2_+ \to H^2_-$ is defined by the formula

$$K(\omega)f = P_-(\omega f). \tag{1.17}$$

In this context, ω is called the *symbol* of $K(\omega)$. The definition (1.17) makes sense, for example, on all polynomials f. It is evident that $K(\omega)$ depends only on the part $P_-\omega$ of ω, i.e. $K(\omega) = K(P_-\omega)$. Nehari's theorem ensures that $K(\omega)$ is a bounded operator if and only if $P_-\omega \in \mathrm{BMO}$, and Hartman's theorem says that $K(\omega)$ is compact if and only if $P_-\omega \in \mathrm{VMO}$; see Proposition 2.4 below.

Another equivalent point of view on Hankel operators appears when one considers the matrix representation of $K(\omega)$ with respect to the standard bases in $H^2_\pm$. Consider the bases $\{\mu^j\}_{j=0}^\infty$ in H^2_+ and $\{\mu^{-1-k}\}_{k=0}^\infty$ in H^2_-. Then the matrix representation of $K(\omega)$ with respect to this pair of bases is

$$(K(\omega)\mu^j, \mu^{-1-k})_{L^2} = \widehat{\omega}(-1-j-k), \quad j,k \geq 0.$$

It will be convenient to have a separate piece of notation for such infinite matrices, considered as operators on the sequence space $\ell^2 := \ell^2(\mathbb{Z}_+)$. Given a sequence $\{h(j)\}_{j=0}^\infty$ of complex numbers, we define the Hankel operator $\Gamma(h)$ on ℓ^2 by

$$(\Gamma(h)u)(j) = \sum_{k=0}^\infty h(j+k)u(k). \tag{1.18}$$

Now suppose $\omega \in L^2$ and $P_-\omega \in \mathrm{VMO}$; take $h(j) = \widehat{\omega}(-1-j)$ for all $j \geq 0$. Then the operators $K(\omega)$ and $\Gamma(h)$ have the same matrix representation with respect to some pairs of orthonormal bases, and hence $\Gamma(h) = \mathcal{U}_-^* K(\omega)\mathcal{U}_+$ for appropriate unitary mappings $\mathcal{U}_\pm : \ell^2 \to H^2_\pm$. It follows that these operators have the same sequence of singular values (see Section 2.1):

$$s_n(K(\omega)) = s_n(\Gamma(h)), \quad \forall n \geq 0, \quad \text{if} \quad h(j) = \widehat{\omega}(-1-j), \quad \forall j \geq 0.$$

The proof of our main result relies on the following two ingredients:

- The Adamyan-Arov-Krein (AAK) theorem. One of the alternative ways to state this theorem is to say that

  $$s_n(K(\omega)) = \rho_n^-(\omega), \quad n \geq 0,$$

  if $P_-\omega \in \mathrm{VMO}$. We give some background related to this formula in Section 2.
- Our results of [19, 20], which give an asymptotic formula for the singular values of a class of Hankel operators $\Gamma(h)$. Those are the operators corresponding to the sequences h of the form

  $$h(j) = j^{-1}(\log j)^{-\alpha} + \text{error term}, \quad j \to \infty, \tag{1.19}$$

  and, more generally,

  $$h(j) = \sum_{\ell=1}^L b_\ell j^{-1}(\log j)^{-\alpha}\zeta_\ell^{-j} + \text{error term}, \quad j \to \infty, \tag{1.20}$$

  where $b_1, \dots, b_L \in \mathbb{C}$ and $\zeta_1, \dots, \zeta_L \in \mathbb{T}$. Sequences of the type (1.19) are required in the proof of Theorems 1.1 and 1.3, while sequences of the type (1.20) are required in the proof of the more general Theorems 3.5 and 3.8, which pertain to the functions ω with several ($= L$) singularities on the unit circle.

Our construction depends on the interplay between two representations of Hankel operators: as $K(\omega) : H^2_+ \to H^2_-$ and as $\Gamma(h) : \ell^2 \to \ell^2$. From the technical point of view, we only have to relate the class of Hankel operators $K(\omega)$, where the symbol ω has finitely many ($= L$) logarithmic singularities on $\mathbb{T}$, to the class of Hankel operators $\Gamma(h)$, where h is of the form (1.20). This requires a rather careful analysis of the Fourier coefficients of such functions ω. We show that every singularity of ω generates one of the terms in the right-hand side of (1.20). This result is stated as Theorem 3.2.

1.6. **The structure of the paper.** In Section 2 we recall some background information related to the theory of Hankel operators and to the AAK theorem. In Section 3 we state our main results, Theorems 3.5 and 3.8, which are extensions of Theorems 1.1 and 1.3. In the same Section, we deduce our main results from the technical Theorem 3.2, which describes the asymptotic behaviour of the Fourier coefficients of functions with logarithmic singularities on $\mathbb{T}$. The proof of Theorem 3.2 is given in Section 4.

2. Background information

2.1. **Schatten classes.** Here we briefly recall some background facts on Schatten classes; for a detailed presentation, see, e.g. the book [4]. Let $\mathcal{B}$ be the algebra of bounded operators on a Hilbert space $\mathcal{H}$, and let $\|\cdot\|$ be the operator norm. Singular values of a compact operator $A \in \mathcal{B}$ are defined by the relation $s_n(A) = \lambda_n(|A|)$, where $\{\lambda_n(|A|)\}_{n=0}^\infty$ is the non-increasing sequence of eigenvalues of the compact non-negative operator $|A| = \sqrt{A^*A}$ (with multiplicities taken into account). Singular values may also be defined by the relation

$$s_n(A) = \min\{\|A - B\| : B \in \mathcal{B}, \quad \operatorname{rank} B \le n\}, \quad n = 0, 1, \dots. \tag{2.1}$$

For $p > 0$, the Schatten class $\mathbf{S}_p$ and the weak Schatten class $\mathbf{S}_{p,\infty}$ of compact operators are defined by the conditions

$$\begin{aligned} A \in \mathbf{S}_p \quad &\Leftrightarrow \quad \sum_{n=0}^\infty s_n(A)^p < \infty, \\ A \in \mathbf{S}_{p,\infty} \quad &\Leftrightarrow \quad \sup_{n\ge 0} (n+1)^{1/p} s_n(A) < \infty. \end{aligned}$$

Of course, we have $\mathbf{S}_p \subset \mathbf{S}_{p,\infty}$.

2.2. **Relations between $\rho_n(\omega)$, $\rho_n^+(\omega)$, $\rho_n^-(\omega)$.** Recall that $\mathcal{R}_n$ consists of all rational functions with at most n poles, including the pole at infinity, but with no poles on the unit circle $\mathbb{T}$; the poles are counted with multiplicities taken into account. Thus, $r \in \mathcal{R}_n$ if and only if

$$r(z) = p(z) + \sum_{|z_j| \ne 1} \sum_{\ell=1}^{L_j} c_{j,\ell} (z - z_j)^{-\ell},$$

where p is a polynomial and

$$\deg r = \deg p + L_1 + L_2 + \cdots \leq n.$$

Hence the functions $r_+ = P_+ r \in \mathcal{R}_n^+$ and $r_- = P_- r \in \mathcal{R}_n^-$ are given by

$$r_+(z) = p(z) + \sum_{|z_j|>1} \sum_{\ell=1}^{L_j} c_{j,\ell}(z-z_j)^{-\ell}, \quad r_-(z) = \sum_{|z_j|<1} \sum_{\ell=1}^{L_j} c_{j,\ell}(z-z_j)^{-\ell}. \quad (2.2)$$

The following simple relations between the distances $\rho_n(\omega)$, $\rho_n^+(\omega)$ and $\rho_n^-(\omega)$ (see (1.9)) will be useful.

Lemma 2.1. *For any* $\omega \in \mathrm{VMO}$ *and any* $n \geq 0$*, we have the relation*

$$\rho_n^+(\overline{\omega}) = \rho_n^-(\omega). \quad (2.3)$$

Moreover, formula (1.11) *holds true.*

Proof. Put $H_+^2(0) = \{f \in H_+^2 : \widehat{f}(0) = 0\}$ and $\mathcal{R}_n^+(0) = \{r \in \mathcal{R}_n^+ : \widehat{r}(0) = 0\}$. Then $\overline{H_+^2(0)} = H_-^2$, $\overline{\mathcal{R}_n^+(0)} = \mathcal{R}_n^-$ and $H_+^2 = H_+^2(0) + \mathbb{C}$, $\mathcal{R}_n^+ = \mathcal{R}_n^+(0) + \mathbb{C}$. Therefore, by the definition (1.9) of ρ_n^+, we have

$$\rho_n^+(\overline{\omega}) = \min\{\|P_+(\overline{\omega} - r_+ - r_0)\|_{\mathrm{BMO}} : r_+ \in \mathcal{R}_n^+(0), r_0 \in \mathbb{C}\}$$

which in view of the relation (1.7) yields

$$\rho_n^+(\overline{\omega}) = \max\big\{\min\{\|P_-\omega - r_-\|_{\mathrm{BMO}} : r_- \in \mathcal{R}_n^-\}, \min\{|\widehat{\omega}(0) - r_0| : r_0 \in \mathbb{C}\}\big\}.$$

Choosing $r_0 = \widehat{\omega}(0)$, we see that the right-hand side here equals $\rho_n^-(\omega)$.

Putting together relations (1.10) and (2.3) and passing to the complex conjugation, we see that

$$\rho_n^+(\omega) = \mathrm{dist}_{L^\infty}\{\overline{\omega}, \mathcal{R}_n^- + H_+^2\} = \mathrm{dist}_{L^\infty}\{\omega, \overline{\mathcal{R}_n^-} + \overline{H_+^2}\}.$$

Since

$$\overline{\mathcal{R}_n^-} + \overline{H_+^2} = \mathcal{R}_n^+(0) + \overline{H_+^2(0) + \mathbb{C}} = \mathcal{R}_n^+(0) + H_-^2 + \mathbb{C},$$

we obtain formula (1.11). □

Lemma 2.2. *For any* $\omega \in \mathrm{VMO}$ *and any* $n \geq 0$*, we have the relation*

$$\rho_n(\omega) = \min\big\{\max\{\rho_{n_+}^+(\omega), \rho_{n_-}^-(\omega)\} : n_+ + n_- = n\big\}. \quad (2.4)$$

Proof. For $r \in \mathcal{R}_n$, denote $r_\pm = P_\pm r$. From (2.2), it is easy to see that $r_+ \in \mathcal{R}_{n_+}^+$ and $r_- \in \mathcal{R}_{n_-}^-$, where $n_+ + n_- = n$. Conversely, if $r_\pm \in \mathcal{R}_{n_\pm}^\pm$, then $r = r_+ + r_- \in \mathcal{R}_n$ with $n = n_+ + n_-$. By the identity (1.8), we have

$$\|\omega - r\|_{\mathrm{BMO}} = \max\{\|P_+\omega - r_+\|_{\mathrm{BMO}}, \|P_-\omega - r_-\|_{\mathrm{BMO}}\}.$$

It follows that

$$\rho_n(\omega) = \min\big\{\max\{\|P_+\omega - r_+\|_{\mathrm{BMO}}, \|P_-\omega - r_-\|_{\mathrm{BMO}}\} : r_\pm \in \mathcal{R}_{n_\pm}^\pm, n_+ + n_- = n\big\};$$

the right-hand side here coincides with the right-hand side in (2.4). □

It will be convenient to rewrite (2.4) in terms of the following counting functions:

$$\nu(\omega; s) = \#\{n \geq 0 : \rho_n(\omega) > s\}, \quad \nu^{\pm}(\omega; s) = \#\{n \geq 0 : \rho_n^{\pm}(\omega) > s\}, \tag{2.5}$$

where $s > 0$.

Lemma 2.3. *For any $\omega \in \mathrm{VMO}$ and any $s > 0$, the relation*

$$\nu(\omega; s) = \nu^{+}(\omega; s) + \nu^{-}(\omega; s) \tag{2.6}$$

holds true.

Proof. Fix $s > 0$. Observe that $\rho_n(\omega) \leq s$ is equivalent to $\nu(\omega; s) \leq n$, and similarly for $\rho_n^{\pm}(\omega)$. By (2.4), for any $n \geq 0$, the relation $\rho_n(\omega) \leq s$ is equivalent to

$$\exists n_+, n_- : \quad n_+ + n_- = n, \quad \rho_{n_+}^{+}(\omega) \leq s \text{ and } \rho_{n_-}^{-}(\omega) \leq s.$$

This can be rewritten as

$$\exists n_+, n_- : \quad n_+ + n_- = n, \quad \nu^{+}(\omega; s) \leq n_+ \text{ and } \nu^{-}(\omega; s) \leq n_-,$$

which is equivalent to $\nu^{+}(\omega; s) + \nu^{-}(\omega; s) \leq n$. We have proven that $\nu(\omega; s) \leq n$ is equivalent to $\nu^{+}(\omega; s) + \nu^{-}(\omega; s) \leq n$; thus, we get (2.6). □

2.3. **Hankel operators on Hardy spaces.** Here we recall several fundamental results of the theory of Hankel operators. The first proposition below is Nehari's theorem [13], which we combine for convenience with the compactness result due to P. Hartman [10].

Proposition 2.4. [16, Theorems 1.1.3 and 1.5.8] *Suppose that $\omega \in L^2$. Then the Hankel operator $K(\omega) : H^2_+ \to H^2_-$ is bounded (resp. compact) if and only if $P_-\omega \in \mathrm{BMO}$ (resp. $P_-\omega \in \mathrm{VMO}$). Moreover,*

$$\|K(\omega)\| = \|P_-\omega\|_{\mathrm{BMO}}. \tag{2.7}$$

In view of Proposition 2.4, the definition (1.6) of the BMO norm can be rewritten as

$$\|\omega\|_{\mathrm{BMO}} = \max\{\|K(\omega)\|, \|K(\overline{\omega})\|, |\widehat{\omega}(0)|\}.$$

The Kronecker theorem describes all finite rank Hankel operators.

Proposition 2.5. *A Hankel operator $K(\omega)$ has rank n if and only if $P_-\omega \in \mathcal{R}_n$ (equivalently, if and only if $P_-\omega \in \mathcal{R}_n^-$).*

The Adamyan-Arov-Krein theorem states that, for Hankel operators, the minimum in (2.1) can be taken over Hankel operators only. We denote by $\mathcal{K}$ the set of all bounded Hankel operators.

Proposition 2.6. [1, Theorem 0.1] *Let K be a compact Hankel operator. Then*

$$s_n(K) = \min\{\|K - G\| : G \in \mathcal{K}, \operatorname{rank} G \leq n\}, \quad n = 0, 1, \ldots.$$

Combining Kronecker and Adamyan-Arov-Krein theorems and taking into account relation (2.7) and Lemma 2.1, one obtains the following result (which is essentially Theorem 0.2 in [1]).

Proposition 2.7. *Let $\omega \in \mathrm{VMO}$. Then for all $n \geq 0$,*

$$\rho_n^+(\omega) = s_n(K(\overline{\omega})), \quad \rho_n^-(\omega) = s_n(K(\omega)).$$

Thus, the problem of rational approximation of a function $\omega \in \mathrm{VMO}$ is equivalent to the study of the singular values of the corresponding Hankel operator.

2.4. **Schatten class properties of Hankel operators.** Here we recall important results due to V. Peller that characterise Hankel operators of Schatten classes. Some partial results in this direction were independently obtained by S. Semmes in [21] and by A. A. Pekarskiĭ in [17]. Definitions of the Besov class $B^{1/p}_{p,p}$ and the Besov-Lorentz classe $\mathfrak{B}^{1/p}_{p,\infty}$ are given in the Appendix; further relevant information can be found in Peller's book [16]. We will not need the three propositions below in our construction, and they are given here only in order to put our results into the right context.

Proposition 2.8. [16, Corollaries 6.1.2, 6.2.2 and 6.3.2] *Let $\omega \in L^2$ and $p > 0$. Then the Hankel operator $K(\omega)$ belongs to the Schatten class $\mathbf{S}_p$ if and only if $P_-\omega \in B^{1/p}_{p,p}$.*

In view of Proposition 2.6, this implies the following result on rational approximation in the BMO norm.

Proposition 2.9. [16, Theorem 6.6.1] *Let $\omega \in \mathrm{VMO}$ and $p > 0$. Then the condition*

$$\big\{\mathrm{dist}_{\mathrm{BMO}}\{\omega, \mathcal{R}_n\}\big\}_{n=0}^{\infty} \in \ell^p(\mathbb{Z}_+)$$

is satisfied if and only if $\omega \in B^{1/p}_{p,p}$.

Using real interpolation, Peller has also obtained the "weak version" of this result (see [16, Section 6.4]).

Proposition 2.10. *Let $\omega \in \mathrm{VMO}$ and $p > 0$. Then the condition*

$$\mathrm{dist}_{\mathrm{BMO}}\{\omega, \mathcal{R}_n\} = O(n^{-1/p}), \quad n \to \infty,$$

is satisfied if and only if $\omega \in \mathfrak{B}^{1/p}_{p,\infty}$.

2.5. **Hankel operators in ℓ^2.** Here we state the result of [20] on the asymptotics of singular values of Hankel operators $\Gamma(h)$. First we need some notation. For a sequence $g = \{g(j)\}_{j=0}^{\infty}$, we define iteratively the sequences $g^{(m)} = \{g^{(m)}(j)\}_{j=0}^{\infty}$, $m = 0, 1, 2, \dots$, by setting $g^{(0)}(j) = g(j)$ and

$$g^{(m+1)}(j) = g^{(m)}(j+1) - g^{(m)}(j), \quad j \geq 0.$$

Theorem 2.11. [20, Theorem 3.1] *Let $\alpha > 0$, let $\zeta_1, \zeta_2, \dots, \zeta_L \in \mathbb{T}$ be distinct numbers, and let $b_1, b_2, \dots, b_L \in \mathbb{C}$. Let h be a sequence of complex numbers such that*

$$h(j) = \sum_{\ell=1}^{L} \big(b_\ell j^{-1}(\log j)^{-\alpha} + g_\ell(j)\big)\zeta_\ell^{j+1}, \quad j \geq 2, \tag{2.8}$$

where the error terms g_ℓ, $\ell = 1, \dots, L$, obey the estimates

$$g_\ell^{(m)}(j) = o(j^{-1-m}(\log j)^{-\alpha}), \quad j \to \infty, \tag{2.9}$$

for all $m = 0, 1, \dots$. Then the Hankel operator $\Gamma(h)$ defined by formula (1.18) *is compact in ℓ^2, and its singular values satisfy the asymptotic relation*

$$s_n(\Gamma(h)) = a\, n^{-\alpha} + o(n^{-\alpha}), \quad a = \varkappa(\alpha)\Big(\sum_{\ell=1}^{L} |b_\ell|^{1/\alpha}\Big)^{\alpha}, \tag{2.10}$$

as $n \to \infty$, where the coefficient $\varkappa(\alpha)$ is given by formula (1.14).

Remark. In fact, it suffices to require condition (2.9) for $0 \leq m \leq M(\alpha)$, where $M(\alpha)$ is an explicit finite number.

3. Main results

The structure of this section is as follows. First, we state a technical Theorem 3.2, which gives the asymptotics of the Fourier coefficients for functions ω with logarithmic singularities on the unit circle. The proof of this theorem will be provided in the next section. Then, using this theorem, we prove Theorem 3.3, which yields the asymptotics of the singular values for Hankel operators $K(\omega)$ with ω as above. Finally, we state and prove our main results (Theorems 3.5 and 3.8) on rational approximation of such functions ω. They are obtained as simple corollaries of Theorem 3.3.

3.1. **Fourier coefficients of singular functions.** Here we consider functions $\omega(\mu)$ which are smooth on the unit circle except at the point $\mu = 1$, where $\omega(\mu)$ have logarithmic singularities. These singularities will be slightly more general than those of the "model functions" ω_0, $\omega_\pm$ of Section 1 (see (1.12), (1.13)) and will contain additional functional parameters.

As in Section 1, we fix an even function $\chi_0 \in C^\infty(\mathbb{R})$ satisfying the condition

$$\chi_0(\theta) = \begin{cases} 1 & \text{for } |\theta| \leq c_1, \\ 0 & \text{for } |\theta| \geq c, \end{cases} \tag{3.1}$$

where $0 < c_1 < c$ are sufficiently small numbers (we will be more specific below). First let us informally discuss the structure of an admissible singularity of ω at the point $\mu = 1$ of the unit circle. Below the index σ takes values $+$ and $-$ and $\mathbb{1}_\sigma$ denotes the characteristic function of the semi-axis $\mathbb{R}_\sigma$. The more general version of ω_0 (see (1.12)) is the function

$$\sum_{\sigma=\pm} v_{0,\sigma}(\theta)(-\log|\theta| + u_{0,\sigma}(\theta))^{1-\alpha}\mathbb{1}_\sigma(\theta)\chi_0(\theta)$$

where $v_{0,\sigma}$ and $u_{0,\sigma}$ are arbitrary complex valued C^∞ functions such that

$$v_{0,+}(0) = v_{0,-}(0) =: v_0. \tag{3.2}$$

Similarly, the generalisation of the linear combination of ω_+ and ω_- (see (1.13)) is

$$\sum_{\sigma=\pm} v_{1,\sigma}(\theta)(-\log|\theta| + u_{1,\sigma}(\theta))^{-\alpha}\mathbb{1}_\sigma(\theta)\chi_0(\theta)$$

with some C^∞ functions $v_{1,\sigma}$ and $u_{1,\sigma}$. Below we combine these two expressions more succinctly as a sum of four terms. More precisely, we introduce the following assumption.

Assumption 3.1. *Let $\alpha > 0$, and let $v_{j,\sigma}(\theta)$ and $u_{j,\sigma}(\theta)$, $j = 0, 1$, $\sigma = \pm$, be complex valued C^∞ functions of $\theta \in \mathbb{R}$ such that condition* (3.2) *is satisfied. Then the function ω is defined by the relation*

$$\omega(e^{i\theta}) = \sum_{j=0,1}\sum_{\sigma=\pm} v_{j,\sigma}(\theta)(-\log|\theta| + u_{j,\sigma}(\theta))^{1-j-\alpha}\mathbb{1}_\sigma(\theta)\chi_0(\theta), \quad \theta \in (-\pi, \pi]. \tag{3.3}$$

Here c_2 is chosen so small that $\theta = 0$ is the only singularity of the functions in the sum (3.3), that is,

$$-\log|\theta| + u_{j,\sigma}(\theta) \neq 0 \quad \text{if} \quad \theta \in [-c_2, c_2]$$

for $j = 0, 1$, $\sigma = \pm$. The function $z^{j-\alpha}$ for $z = -\log|\theta| + u_{j,\sigma}(\theta)$ in (3.3) is defined by the principal branch, $z^{j-\alpha} = e^{(j-\alpha)\log z}$, where we assume that

$$\arg(-\log|\theta| + u_{j,\sigma}(\theta)) \to 0 \quad \text{as} \quad \theta \to 0$$

for all these functions.

For a function ω satisfying Assumption 3.1, we put

$$b = b(\omega) = (1-\alpha)v_0\big(\tfrac{1}{2} - \tfrac{1}{2\pi i}(u_{0,+}(0) - u_{0,-}(0))\big) - \tfrac{1}{2\pi i}(v_{1,+}(0) - v_{1,-}(0)). \tag{3.4}$$

The analytic core of our construction is the following theorem.

Theorem 3.2. *Under Assumption* 3.1, *the Fourier coefficients of $\omega(\mu)$ admit the representation*

$$\widehat{\omega}(-j) = bj^{-1}(\log j)^{-\alpha} + g(-j), \quad j \geq 2, \tag{3.5}$$

where the coefficient $b = b(\omega)$ is given by formula (3.4) *and the error term $g(-j)$ satisfies the estimates*

$$g^{(m)}(-j) = O\big(j^{-1-m}(\log j)^{-\alpha-1}\big), \quad j \to +\infty, \tag{3.6}$$

for all $m \geq 0$.

We emphasize that the leading terms of the asymptotics of the Fourier coefficients of these functions depend on the combination (3.4) only.

The proof of Theorem 3.2 will be given in the next section.

3.2. **Hankel operators with singular symbols.** Here we state a result about the singular value asymptotics for Hankel operators $K(\omega)$ with symbols ω having finitely many logarithmic singularities.

Theorem 3.3. *Let $\zeta_1, \zeta_2, \dots, \zeta_L \in \mathbb{T}$ be distinct numbers, and let the functions $\omega_1, \omega_2, \dots, \omega_L$ satisfy Assumption* 3.1. *Define the function*

$$\omega(\mu) = \sum_{\ell=1}^{L} \omega_\ell(\mu/\zeta_\ell), \quad \mu \in \mathbb{T}, \tag{3.7}$$

and set

$$a(\omega) = \varkappa(\alpha)\Big(\sum_{\ell=1}^{L} |b(\omega_\ell)|^{1/\alpha}\Big)^\alpha, \tag{3.8}$$

where the numbers $b(\omega_\ell)$ are given by (3.4) *and $\varkappa(\alpha)$ is the coefficient* (1.14)*. Then the Hankel operator $K(\omega)$ is compact and its singular values have the asymptotics*

$$s_n(K(\omega)) = a(\omega)\, n^{-\alpha} + o(n^{-\alpha}),$$

as $n \to \infty$.

Proof. Observe that for arbitrary $\zeta \in \mathbb{T}$ and $\phi \in L^1(\mathbb{T})$, we have

$$\widehat{\phi}_\zeta(j) = \widehat{\phi}(j)\zeta^{-j} \quad \text{if} \quad \phi_\zeta(\mu) = \phi(\mu/\zeta).$$

Therefore it follows from (3.7) that

$$\widehat{\omega}(-j-1) = \sum_{\ell=1}^{L} \widehat{\omega}_\ell(-j-1)\zeta_\ell^{j+1}.$$

Let $h(j) = \widehat{\omega}(-j-1)$. According to Theorem 3.2 the sequence $h(j)$ satisfies condition (2.8) as $j \to +\infty$; the corresponding asymptotic coefficients $b_\ell = b(\omega_\ell)$ are defined by formula (3.4). Thus Theorem 2.11 implies the asymptotic formula (2.10) for the singular values of the Hankel operator $\Gamma(h)$. Since $\Gamma(h)$ and $K(\omega)$ have the same set of singular values, we obtain the desired result. $\square$

It is important that the singularities of the symbol (3.7) are located at distinct points $\zeta_1, \dots, \zeta_L$.

Remark 3.4. Let ω be a function satisfying Assumption 3.1, but without the condition (3.2), i.e. with $v_{0,+}(0) \neq v_{0,-}(0)$; simple examples of such function are

$$\omega_\pm(e^{i\theta}) = |\log|\theta||^{1-\alpha}\mathbb{1}_\pm(\theta)\chi_0(\theta) \quad \text{or} \quad \omega(e^{i\theta}) = |\log|\theta||^{1-\alpha}\operatorname{sign}\theta\,\chi_0(\theta).$$

Now the terms with $j = 1$ in (3.3) are inessential and instead of (3.4) we put

$$\widetilde{b}(\omega) = -\tfrac{1}{2\pi i}(v_{0,+}(0) - v_{0,-}(0)).$$

Let ω be given by formula (3.7) where each ω_ℓ is as above. Then the operator $K(\omega)$ is compact for $\alpha > 1$ only, and the asymptotics of its singular values is of a different order:

$$s_n(K(\omega)) = \widetilde{a}\, n^{-\gamma} + o(n^{-\gamma}), \quad \gamma = \alpha - 1,$$

where

$$\widetilde{a} = \varkappa(\gamma)\Big(\sum_{\ell=1}^{L} |b(\widetilde{\omega}_\ell)|^{1/\gamma}\Big)^{\gamma}.$$

This fact follows from Theorem 3.3 with $v_0 = 0$ and α replaced by $\alpha - 1$.

3.3. **Rational approximation.** We recall that the distances $\rho_n(\omega)$ and $\rho_n^{\pm}(\omega)$ are defined by relations (1.9). Our main result on rational approximation is

Theorem 3.5. *Assume the hypothesis of Theorem* 3.3 *and set*

$$a^+ = a(\overline{\omega}), \quad a^- = a(\omega), \quad a = \big((a^+)^{1/\alpha} + (a^-)^{1/\alpha}\big)^{\alpha}.$$

Then

$$\lim_{n\to\infty} n^{\alpha} \rho_n^{\pm}(\omega) = a^{\pm}, \tag{3.9}$$

$$\lim_{n\to\infty} n^{\alpha} \rho_n(\omega) = a. \tag{3.10}$$

Proof. To prove (3.9), it suffices to put together Proposition 2.7 and Theorem 3.3. In order to prove (3.10), we observe that (3.9) can be equivalently rewritten in terms of the counting functions $\nu^{\pm}(\omega; s)$ (see (2.5)) as

$$\lim_{s\to 0} s^{1/\alpha} \nu^{\pm}(\omega; s) = (a^{\pm})^{1/\alpha}.$$

It now follows from Lemma 2.3 that

$$\lim_{s\to 0} s^{1/\alpha} \nu(\omega; s) = (a^+)^{1/\alpha} + (a^-)^{1/\alpha},$$

which is equivalent to (3.10). □

Remark 3.6. Theorem 3.3 automatically extends to symbols ω that include an error term:

$$\omega(\mu) = \sum_{\ell=1}^{L} \omega_\ell(\mu/\zeta_\ell) + \widetilde{\omega}(\mu), \quad \mu \in \mathbb{T}, \tag{3.11}$$

where $\widetilde{\omega}$ is any symbol such that

$$s_n(K(\widetilde{\omega})) = o(n^{-\alpha}), \quad n \to \infty. \tag{3.12}$$

This follows by a standard application of Ky Fan's lemma (see e.g. [7, Section II.2.5]). Condition (3.12) is satisfied, for example, when $P_- \widetilde{\omega} \in B^{\alpha}_{1/\alpha,1/\alpha}$. Therefore Theorem 3.5 is also true for functions (3.11) where $\widetilde{\omega} \in B^{\alpha}_{1/\alpha,1/\alpha}$.

Theorem 1.1 is a particular case of Theorem 3.5, with the following choice of parameters: $L = 1$, $\zeta_1 = 1$, and

$$v_{0,\pm}(\theta) = v_0, \quad v_{1,\pm}(\theta) = v_\pm, \quad u_{0,\pm}(\theta) = 0, \quad u_{1,\pm}(\theta) = 0.$$

Formula (3.8) is quite intuitive from the viewpoint of singular value asymptotics. It means that the contributions of different singularities of the symbol ω to the singular values counting function are independent of each other. On the other hand, this formula does not look obvious in the approximation theory framework.

3.4. **Rational approximation of analytic functions.** Let us consider the case of $\omega(z)$ analytic in the unit disc; then $\rho_n(\omega) = \rho_n^+(\omega)$. Let $u(z)$ be analytic in $\mathbb{D}$, $u \in C^\infty(\overline{\mathbb{D}})$; fix some $\zeta \in \mathbb{T}$ and assume that

$$-\log(\zeta - z) + u(z) \neq 0, \quad z \in \overline{\mathbb{D}}. \tag{3.13}$$

Define

$$\omega(z) = \big(-\log(\zeta - z) + u(z)\big)^{1-\alpha}, \quad z \in \mathbb{D}, \quad \alpha > 0. \tag{3.14}$$

The branch of the analytic function $\log(\zeta - z)$ is fixed by the condition $\log(\zeta - z) = \log(1-r) + i\varphi_0$ if $z = r\zeta$, $r \in (0,1)$, and $\zeta = e^{i\varphi_0}$. We fix $\arg\big(-\log(\zeta - z) + u(z)\big)$ by the condition that it tends to zero as $z = re^{i\varphi_0}$ and $r \to 1-0$. Obviously the function $\omega(z)$ is analytic in the unit disc $\mathbb{D}$ and is smooth up to the boundary $\mathbb{T}$, except at the point $z = \zeta$. Let us find its asymptotic behavior as $z \in \mathbb{T}$ and $z \to \zeta$.

Lemma 3.7. *Let $\mu = e^{i\psi}$, $\zeta = e^{i\psi_0}$ and $\theta := \psi - \psi_0$. Then the function* (3.14) *admits the representation*

$$\omega(\mu) = \big(-\log|\theta| + u_\pm(\theta)\big)^{1-\alpha}, \quad \pm\theta > 0,$$

where $u_\pm$ are C^∞-smooth functions,

$$u_\pm(\theta) = \pm i\pi/2 - i\theta/2 - \log\frac{\sin(\theta/2)}{\theta/2} + u(e^{i(\psi_0+\theta)}) - i\psi_0. \tag{3.15}$$

In particular,

$$u_+(0) - u_-(0) = i\pi. \tag{3.16}$$

Proof. Observe that

$$\begin{aligned}\log(\zeta - \mu) &= \log(e^{i\psi_0} - e^{i\psi}) = \log(e^{i\psi_0}(1 - e^{i\theta}))\\ &= \log(1 - e^{i\theta}) + i\psi_0 = \log(2\sin|\theta/2|) + i\arg(1 - e^{i\theta}) + i\psi_0\end{aligned}$$

and $\arg(1 - e^{i\theta}) = (\mp\pi + \theta)/2$ for $\pm\theta > 0$. Therefore

$$-\log(\zeta - \mu) + u(\mu) = -\log|\theta| + u_\pm(\theta)$$

where $u_\pm(\theta)$ is given by (3.15). □

Below we consider sums of functions (3.14) with variable coefficients. Lemma 3.7 allows us to apply Theorem 3.5 in the special case $v_{1,\pm}(\theta) = 0$.

Theorem 3.8. *Let* $\zeta_1, \zeta_2, \dots, \zeta_L \in \mathbb{T}$ *be distinct points, and let functions* v_ℓ, u_ℓ, $\ell = 1, \dots, L$, *be analytic in* $\mathbb{D}$ *and* $v_\ell, u_\ell \in C^\infty(\overline{\mathbb{D}})$, *and assume that* (3.13) *is satisfied for all* u_ℓ, ζ_ℓ. *Put*

$$\omega(z) = \sum_{\ell=1}^{L} v_\ell(z)\big(-\log(\zeta_\ell - z) + u_\ell(z)\big)^{1-\alpha}, \quad \alpha > 0.$$

Then there exists the limit

$$\lim_{n\to\infty} n^\alpha \rho_n^+(\omega) = |1-\alpha|\varkappa(\alpha)\Big(\sum_{\ell=1}^{L} |v_\ell(\zeta_\ell)|^{1/\alpha}\Big)^\alpha$$

where the coefficient $\varkappa(\alpha)$ *is given by* (1.14).

Proof. It follows from Lemma 3.7 that the function $\omega(z)$ admits representation (3.11) where every function $\omega_\ell(z)$ satisfies Assumption 3.1 with the corresponding functions $v_{1,\pm}(\theta) = 0$. Therefore according to relations (3.4) and (3.16) we have $b(\overline{\omega_\ell}) = (1-\alpha)v_\ell(\zeta_\ell)$. Now we can apply Theorem 3.5; the smooth error term $\widetilde{\omega}$ does not affect the asymptotics — see Remark 3.6. □

4. Fourier transforms of functions with logarithmic singularities

4.1. **Statement of the result.** Our goal in this section is to prove Theorem 3.2. In fact, we prove a slightly more general statement, where Fourier coefficients are replaced by Fourier transforms. It is convenient to introduce the function of $x \in \mathbb{R}$,

$$\Omega(x) = \begin{cases} \omega(e^{ix}) & -\pi < x \le \pi \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Theorem 4.1. *Under Assumption* 3.1, *the Fourier transform* $\widehat{\Omega}(t)$ *of* $\Omega(x)$ *is a* C^∞ *function on* $\mathbb{R}$, *which can be written as*

$$(2\pi)^{-1/2}\widehat{\Omega}(-t) = bt^{-1}(\log t)^{-\alpha} + G(-t), \quad t > 1, \tag{4.2}$$

where $b = b(\omega)$ *is given by* (3.4) *and the error term* $G(t)$ *satisfies the estimates*

$$G^{(m)}(-t) = O\big(t^{-1-m}(\log t)^{-\alpha-1}\big), \quad t \to +\infty, \tag{4.3}$$

for all $m = 0, 1, \dots$.

Theorem 4.1 will be proven in the rest of this section. Assuming this theorem, we can give

Proof of Theorem 3.2. Observe that

$$\widehat{\omega}(-j) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Omega(x) e^{ijx} dx = (2\pi)^{-1/2}\widehat{\Omega}(-j).$$

So the asymptotics (3.5) for the Fourier coefficients $\widehat{\omega}(-j)$ with the error term $g(-j) = G(-j)$ follows from the asymptotics (4.2) for the Fourier transform $\widehat{\Omega}(-t)$.

We only have to check that the estimates (4.3) for the function G and its derivatives yield the estimates (3.6) for the sequence g. This elementary statement follows, for example, from the explicit formula

$$g^{(m)}(-j) = \int_0^1 dt_1 \int_0^1 dt_2 \cdots \int_0^1 dt_m \, G^{(m)}(-j + t_1 + \cdots + t_m),$$

which can be checked by induction in m. $\square$

Theorem 4.1 is proven below through a sequence of steps. In Lemma 4.2 we compute the asymptotics of the Laplace transform of explicit functions with a logarithmic singularity. In Lemma 4.3, we use a contour deformation argument to reduce the question of asymptotics of the Fourier transform to that of the Laplace transform. In Lemma 4.4 we show that the functions $v_{j,\sigma}$ and $u_{j,\sigma}$ in the definition of ω (see Assumption 3.1) can be replaced by their values at zero. The proof of Theorem 4.1 is concluded in Section 4.4.

4.2. **Laplace and Fourier transforms of logarithmic functions.** Let us start with an elementary result on the asymptotic expansion of the Laplace transform.

Lemma 4.2. *Let $\alpha \in \mathbb{R}$, $m \in \mathbb{Z}_+$ and let $c \in (0,1)$. Then*

$$\int_0^c \big(-\log y\big)^{-\alpha} y^m e^{-yt} dy = t^{-1-m} (\log t)^{-\alpha} \times \big(m! + \alpha \Gamma'(m+1)(\log t)^{-1} + O((\log t)^{-2})\big) \quad (4.4)$$

as $t \to +\infty$ where Γ' is the derivative of the Gamma function Γ.

Proof. First, we split the integral (4.4) into the integrals over $(0, t^{-1/2})$ and over $(t^{-1/2}, c)$. Due to the factor e^{-yt} the second integral decays faster than any power of t^{-1}. Making the change of variables $x = yt$, we see that the first integral equals

$$t^{-1-m} (\log t)^{-\alpha} \int_0^{t^{1/2}} \big(1 - \tfrac{\log x}{\log t}\big)^{-\alpha} x^m e^{-x} dx. \quad (4.5)$$

Since $u = -\frac{\log x}{\log t} \geq -1/2$ for $x \leq t^{1/2}$, we can use the estimate

$$\big|(1+u)^{-\alpha} - 1 + \alpha u\big| \leq C u^2, \quad u \geq -1/2. \quad (4.6)$$

Thus the integral (4.5) equals

$$t^{-1-m} (\log t)^{-\alpha} \int_0^{t^{1/2}} (1 + \alpha \tfrac{\log x}{\log t}) \, x^m e^{-x} dx + R(t), \quad (4.7)$$

where the remainder satisfies the estimate

$$|R(t)| \leq C t^{-1-m} (\log t)^{-\alpha-2} \int_0^\infty (\log x)^2 \, x^m e^{-x} dx.$$

The integral in (4.7) can be extended to $\mathbb{R}_+$ and then calculated in terms of the Gamma function. The arising error decays faster than any power of t^{-1} as $t \to +\infty$. This yields (4.4). $\square$

The full asymptotic expansion of the Laplace transform (4.4) is of course well known; see, e.g. Lemma 3 in [6], but the above proof is slightly simpler than that in [6]. Here we need two terms, but the method allows one to easily obtain the full expansion.

Next, we discuss the Fourier transform. Below we suppose that $\arg x > 0$ for $x > 0$. Fix some complex number a. We choose a number $c > 0$ so small that $-\log x + a \neq 0$ for $x \in (0, c)$.

Lemma 4.3. *Let $a \in \mathbb{C}$, $\alpha \in \mathbb{R}$ and $m \in \mathbb{Z}_+$. Suppose that a C^∞ function χ_0 satisfies condition* (3.1) *with a sufficiently small c. Then*

$$\int_{-\infty}^{\infty} (-\log|x| + a)^{-\alpha} \mathbb{1}_\pm(x)\chi_0(x)x^m e^{ixt} dx = \pm i^{m+1} t^{-m-1} (\log t)^{-\alpha} \times \Big(m! + \alpha\big(\Gamma'(m+1) + m!(\pm\pi i/2 - a)\big)(\log t)^{-1} + O((\log t)^{-2})\Big) \quad (4.8)$$

as $t \to +\infty$.

Proof. Consider first the sign " + ". For $x \in (0, c)$, denote

$$A(x,t) = \int_0^x \big(-\log z + a\big)^{-\alpha} z^m e^{izt} dz. \quad (4.9)$$

Integrating by parts, we see that

$$\int_0^\infty \big(-\log x + a\big)^{-\alpha} \chi_0(x) x^m e^{ixt} dx = -\int_0^\infty A(x,t)\chi_0'(x) dx \quad (4.10)$$

where $\chi_0' \in C_0^\infty(\mathbb{R}_+)$. Our plan is to find the asymptotics of $A(x,t)$ as $t \to +\infty$ for x in compact subsets of $(0,c)$ and to substitute it into (4.10). Let us choose $\kappa > 0$ so small that $-\log z + a \neq 0$ for z in the closed rectangle in the complex plane with the vertices $0, i\kappa, i\kappa + c, c$. Instead of $(0,x)$, we can integrate over the line segments $(0, i\kappa)$, $(i\kappa, i\kappa + x)$ and $(i\kappa + x, x)$ in (4.9):

$$A(x,t) = \left(\int_0^{i\kappa} + \int_{i\kappa}^{i\kappa+x} + \int_{i\kappa+x}^{x}\right)(-\log z + a)^{-\alpha} z^m e^{izt} dz =: A_0(t) + A_1(x,t) + A_2(x,t).$$

Let us first consider the integral over $(0, i\kappa)$. Setting $z = iy$ and using (4.6), we see that

$$A_0(t) = i^{m+1} \int_0^\kappa (-\log y - i\pi/2 + a)^{-\alpha} y^m e^{-yt} dy$$
$$= i^{m+1} \int_0^\kappa (-\log y)^{-\alpha} \big(1 + (i\pi/2 - a)\alpha(-\log y)^{-1} + \varepsilon(y)\big) y^m e^{-yt} dy$$

where $\varepsilon(y) = O((\log y)^{-2})$ as $y \to 0$. Thus we have reduced the question to computing the asymptotics of the Laplace transform. Now it follows from formula (4.4) that

$$A_0(t) = i^{m+1} t^{-1-m} (\log t)^{-\alpha}$$
$$\times \big(m! + \alpha(\Gamma'(m+1) + m!(i\pi/2 - a))(\log t)^{-1} + O((\log t)^{-2})\big).$$

Substituting this asymptotics into the integral (4.10), we get the right-hand side of (4.8).

It remains to show that the terms A_1 and A_2 do not contribute to the asymptotics of the integral (4.10). Making the change of variables $z = i\kappa + y$, we see that the integral

$$A_1(x,t) = \int_{i\kappa}^{i\kappa + x} \big(-\log z + a\big)^{-\alpha} z^m e^{izt} dz$$
$$= e^{-\kappa t} \int_0^x \big(-\log(i\kappa + y) + a\big)^{-\alpha} (i\kappa + y)^m e^{iyt} dy$$

decays exponentially as $t \to \infty$. This implies that the contribution of $A_1(x,t)$ to the integral (4.10) also decays exponentially.

Similarly, making the change of variables $z = x + i\kappa y$, we can rewrite A_2 as

$$A_2(x,t) = \int_{i\kappa + x}^{x} \big(-\log z + a\big)^{-\alpha} z^m e^{izt} dz$$
$$= -i\kappa e^{itx} \int_0^1 \big(-\log(x + i\kappa y) + a\big)^{-\alpha} (x + i\kappa y)^m e^{-\kappa y t} dy.$$

In the right-hand side we can integrate by parts arbitrarily many times. It is important that the integrand $\big(-\log(x + i\kappa y) + a\big)^{-\alpha}(x + i\kappa y)^m$ is a C^∞-smooth function of $y \in [0,1]$. The contribution of the point $y = 1$ decays exponentially and therefore we have the asymptotic expansion

$$A_2(x,t) = e^{itx} \sum_{n=1}^{N} a_n(x) t^{-n} + O(t^{-N-1}), \quad t \to +\infty, \quad \forall N > 0, \tag{4.11}$$

with some functions $a_n(x)$ that are smooth on the interval $(0, c)$. Substituting (4.11) into (4.10) and integrating by parts with respect to x, we see that the contribution of $A_2(x,t)$ decays faster than any power of t^{-1} as $t \to \infty$.

To prove (4.8) for the sign "$-$", we take the relation (4.8) for the sign "$+$", make the change of the variables $x \mapsto -x$ and pass to the complex conjugation. $\square$

For $a = 0$, the asymptotics of the oscillating integral (4.8) is well known (see [24]), although our proof seems to be somewhat simpler than that in [24] even in this case. So we have given the proof of Lemma 4.3 mainly for the completeness of our presentation. Of course the method of proof of Lemma 4.3 yields the complete asymptotic expansion of the integral (4.8), but we do not need it.

4.3. **Replacing $v(x)$ by $v(0)$ and $u(x)$ by $u(0)$.** Here our aim is to prove that the variable parameters $v_{j,\sigma}$ and $u_{j,\sigma}$ in the definition of the function ω (see Assumption 3.1) can be replaced by their values at zero.

Lemma 4.4. *Let $\alpha \in \mathbb{R}$, $m \in \mathbb{Z}_+$, $\sigma = \pm$, and let functions $v, u \in C^\infty$. Then*

$$\int_{-\infty}^{\infty} \Big(v(x)(-\log|x| + u(x))^{-\alpha} - v(0)(-\log|x| + u(0))^{-\alpha}\Big) \mathbb{1}_\sigma(x)\chi_0(x) x^m e^{ixt} dx = O(t^{-\rho}), \quad \forall \rho < m+2, \quad (4.12)$$

as $t \to \infty$.

Proof. Let $\varphi \in C^{m+1}$ and $\varphi^{(m+2)} \in L^1_{\rm loc}$. Integrating by parts $m+2$ times, we see that

$$\int_{-\infty}^{\infty} \varphi(x)\mathbb{1}_\sigma(x)\chi_0(x)e^{ixt}dx = O(t^{-m-2}) \quad \text{if} \quad \varphi(0) = \cdots = \varphi^{(m+1)}(0) = 0. \quad (4.13)$$

Let us use the fact that $(1+z)^{-\alpha} = 1 - \alpha z + R(z)$ where the remainder

$$R \in C^\infty \quad \text{and} \quad R(0) = R'(0) = 0. \quad (4.14)$$

Therefore we have

$$(-\log|x| + u(x))^{-\alpha} = (-\log|x| + u(0))^{-\alpha}\big(1 + w(x)\big)^{-\alpha} = (-\log|x| + u(0))^{-\alpha}\Big(1 - \alpha w(x) + R(w(x))\Big) \quad (4.15)$$

where

$$w(x) = (-\log|x| + u(0))^{-1}(u(x) - u(0)). \quad (4.16)$$

We substitute the expression (4.15) into the integral (4.12) and consider every term separately.

First we consider

$$v(x)(-\log|x| + u(0))^{-\alpha}x^m = (v(0) + v'(0)x + v_1(x))(-\log|x| + u(0))^{-\alpha}x^m$$

where $v_1 \in C^\infty$ and $v_1(0) = v_1'(0) = 0$. The term with $v(0)$ is cancelled out by the second term in the integrand in (4.12). According to Lemma 4.3, the contribution of $v'(0)x$ is bounded by $Ct^{-m-2}(\log t)^{-\alpha}$, and according to (4.13) the contribution of $v_1(x)$ is bounded by Ct^{-m-2}.

Next, we consider the term

$$v(x)(-\log|x|+u(0))^{-\alpha-1}(u(x)-u(0))x^m.$$

We have $v(x)(u(x)-u(0))x^m = v(0)u'(0)x^{m+1}+R_1(x)$ where $R_1 \in C^\infty$ and $R_1(0) = \cdots = R_1^{(m+1)}(0) = 0$. As we have already seen, by Lemma 4.3 the contribution of $v(0)u'(0)x^{m+1}$ is bounded by $Ct^{-m-2}(\log t)^{-\alpha-1}$, and by (4.13) the contribution of $R_1(x)$ is bounded by Ct^{-m-2}.

It remains to consider the function

$$\varphi(x) = v(x)(-\log|x|+u(0))^{-\alpha}R(w(x))x^m. \tag{4.17}$$

Clearly, $w \in C^\infty((-c,c)\setminus\{0\})$ and differentiating (4.16) we see that $w^{(k)}(x) = O(|x|^{1-k})$ as $x \to 0$ for all $k = 0,1,\dots$. Therefore differentiating the composite function $R(w(x))$ and taking into account (4.14), we find that

$$\frac{d^k}{dx^k}R(w(x)) = O(|x|^{2-k}), \quad k = 0,1,\dots.$$

Finally, differentiating the product (4.17), we see that $\varphi \in C^\infty((-c,c)\setminus\{0\})$ and

$$\varphi^{(k)}(x) = O((-\log|x|)^{-\alpha}|x|^{2+m-k}), \quad k = 0,1,\dots,$$

as $x \to 0$. Thus we can apply relation (4.13) to the function (4.17). $\square$

Putting together Lemmas 4.3 and 4.4, we obtain the following result.

Lemma 4.5. *Let the assumptions of Lemma 4.4 be satisfied. Then, as $t \to +\infty$,*

$$\int_{-\infty}^{\infty} v(x)(-\log|x|+u(x))^{-\alpha}\mathbb{1}_\pm(x)\chi_0(x)x^m e^{ixt}dx = \pm i^{m+1}t^{-m-1}(\log t)^{-\alpha}$$
$$\times\, v(0)\Big(m! + \alpha\big(\Gamma'(m+1)+m!(\pm\pi i/2-u(0))\big)(\log t)^{-1} + O((\log t)^{-2})\Big). \tag{4.18}$$

4.4. **Proof of Theorem 4.1.** Since

$$\left(\frac{d}{dt}\right)^m\big(t^{-1}(\log t)^{-\alpha}\big) = (-1)^m m! t^{-1-m}(\log t)^{-\alpha}\big(1+O((\log t)^{-1})\big),$$

we need to prove that

$$\int_{-\infty}^{\infty}\Omega(x)x^m e^{ixt}dx = 2\pi b i^m m! t^{-1-m}(\log t)^{-\alpha}\big(1+O((\log t)^{-1})\big) \tag{4.19}$$

as $t \to +\infty$, where the asymptotic coefficient b is given by equality (3.4).

Recall that the function $\Omega(x)$ is defined by formulas (3.3) and (4.1). So we only have to substitute this expression for $\Omega(x)$ into the left-hand side and to use Lemma 4.5. Thus, keeping only the leading term in the right-hand side of (4.18), we find that

$$\sum_{\sigma=\pm}\int_{-\infty}^{\infty} v_{1,\sigma}(x)(-\log|x|+u_{1,\sigma}(x))^{-\alpha}\mathbb{1}_\sigma(x)\chi_0(x)x^m e^{ixt}dx$$
$$= i^{m+1}m!(v_{1,+}(0)-v_{1,-}(0))t^{-m-1}(\log t)^{-\alpha}\big(1+(\log t)^{-1}\big). \quad (4.20)$$

Similarly, using (4.18) with α replaced by $\alpha-1$, we obtain

$$\sum_{\sigma=\pm}\int_{-\infty}^{\infty} v_{0,\sigma}(x)(-\log|x|+u_{0,\sigma}(x))^{1-\alpha}\mathbb{1}_\sigma(x)\chi_0(x)x^m e^{ixt}dx = i^{m+1}t^{-m-1}(\log t)^{-\alpha}$$
$$\times\Big\{v_{0,+}(0)\Big(m!\log t+(\alpha-1)\big(\Gamma'(m+1)+m!(\pi i/2-u_{0,+}(0))\big)\Big)$$
$$-v_{0,-}(0)\Big(m!\log t+(\alpha-1)\big(\Gamma'(m+1)+m!(-\pi i/2-u_{0,-}(0))\big)\Big)+O((\log t)^{-1})\Big\}. \quad (4.21)$$

Taking into account condition (3.2), we see that the right-hand side here equals

$$i^{m+1}m!t^{-m-1}(\log t)^{-\alpha}v_0(\alpha-1)(\pi i-u_{0,+}(0)+u_{0,-}(0))\big(1+O((\log t)^{-1})\big).$$

Thus putting together relations (4.20), (4.21), we conclude the proof of (4.19) and hence of Theorem 4.1.

Appendix A. Besov and Besov-Lorentz spaces

Here for completeness we recall the definitions of the Besov class $B^{1/p}_{p,p}$ and the Besov-Lorentz class $\mathfrak{B}^{1/p}_{p,\infty}$ on $\mathbb{T}$. The parameter $p>0$ is arbitrary. We refer to the books [16] (see Section 6.4 and Appendix 2) and [23] for more details.

Let $w\in C_0^\infty(\mathbb{R})$ be a function with the properties $w\geq 0$, $\operatorname{supp} w=[1/2,2]$ and

$$\sum_{n=0}^{\infty} w(t/2^n)=1, \quad \forall t\geq 1.$$

Let f be a distribution on $L^1(\mathbb{T})$ with the Fourier coefficients $\widehat{f}(j)$, $j\in\mathbb{Z}$. For $n\in\mathbb{Z}$, let us denote by f_n the polynomial

$$f_n(\mu)=\sum_{j\in\mathbb{Z}} w(\pm j/2^{|n|})\widehat{f}(j)\mu^j, \quad \mu\in\mathbb{T}, \quad \pm n>0,$$

and let $f_0(\mu)=\widehat{f}(1)\mu+\widehat{f}(0)+\widehat{f}(-1)\overline{\mu}$. The Besov class $B^{1/p}_{p,p}$ is defined by the condition

$$\sum_{n\in\mathbb{Z}} 2^{|n|}\|f_n\|^p_{L^p}<\infty. \quad (A.1)$$

By definition, $f \in \mathfrak{B}^{1/p}_{p,\infty}$ if and only if

$$\sup_{t>0} t^p \sum_{n\in\mathbb{Z}} 2^{|n|} m(\{\mu \in \mathbb{T} : |f_n(\mu)| > t\}) < \infty$$

which is the "weak version" of the condition (A.1). We have

$$B^{1/p}_{p,p} \subset \mathfrak{B}^{1/p}_{p,\infty} \subset B^{1/q}_{q,q}, \quad \forall q > p.$$

The Hölder-Zygmund class Λ_α, $\alpha > 0$, is defined in terms of the difference operator

$$(\Delta_\tau f)(\mu) = f(\tau\mu) - f(\mu), \quad \tau \in \mathbb{T}.$$

By definition, $f \in \Lambda_\alpha$ if and only if

$$\|(\Delta^n_\tau f)(\mu)\|_{L^\infty} \le C|\tau - 1|^\alpha$$

where n is an arbitrary integer such that $n > \alpha$. Observe that Λ_α coincides with the Hölder class C^α if α is not integer and $C^\alpha \subset \Lambda_\alpha$ if α is an integer. We also note that $\Lambda_\alpha \subset \mathfrak{B}^\alpha_{1/\alpha,\infty}$.

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