

# A NONLOCAL DIFFUSION PROBLEM ON MANIFOLDS

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ABSTRACT. In this paper we study a nonlocal diffusion problem on a manifold. These kind of equations can model diffusions when there are long range effects and have been widely studied in Euclidean space. We first prove existence and uniqueness of solutions and a comparison principle. Then, for a convenient rescaling we prove that the operator under consideration converges to a multiple of the usual Heat-Beltrami operator on the manifold. Next, we look at the long time behavior on compact manifolds by studying the spectral properties of the operator. Finally, for the model case of hyperbolic space we study the long time asymptotics and find a different and interesting behavior.

## 1. INTRODUCTION.

In this paper we present a nonlocal diffusion equation set on a manifold and study the properties of its solutions. More precisely, let  $M$  be an  $N$ -dimensional, Riemannian manifold without boundary. We assume that there exists a family of isometries  $\tau_x : M \rightarrow M$  such that  $\tau_x(x) = O$  (where we have denoted by  $O$  a fixed point in  $M$ ). Let  $d\mu$  be a measure on  $M$ , invariant under the family of isometries  $\tau_x$ . Denote by  $s_{xy}$  the geodesic distance between the points  $x$  and  $y$ . For  $J : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  normalized by

$$(1.1) \quad \int_M J(s_{xO}) d\mu_x = 1,$$

we define the operator

$$Lu(x) := \int_M J(s_{xy})u(y) d\mu_y - u(x),$$

and consider the corresponding Cauchy problem

$$(1.2) \quad \begin{cases} u_t - Lu = 0 & \text{in } M \times (0, T], \\ u(x, 0) = u_0(x) & \text{in } M. \end{cases}$$

At some points of the article we will assume further that  $J \in C_0^\infty(\mathbb{R}_{\geq 0})$ .

These kind of diffusion problems have been widely studied in Euclidean space as they model dispersion when there are long range effects (see for instance, [5, 9, 16]). Classical diffusion problems can be seen as infinitesimal limits of these nonlocal diffusion models. This is a very well studied fact from a probabilistic point of view (see, for instance [15, 6]) and has also a PDE counterpart [12, 11].

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Other nonlocal models correspond to singular kernels and have also been widely studied (we cite here [13, 30], for instance, but the literature is huge). This study is being extended to the manifold case, especially on non-compact manifolds such as hyperbolic space  $\mathbb{H}^N$  ([4, 3]). Nevertheless, we will not consider singular kernels in this article.

In Euclidean space, the relation between jump diffusions as in (1.2) and the classical heat equation appears in two different asymptotic limits. On one hand, as we have referred to above, in the infinitesimal limit as jumps go to zero, on the other, as the large time asymptotics. In fact, when stated in  $\mathbb{R}^N$  with  $u_0 \in L^1 \cap L^\infty$ , the solution of (1.2) decays as  $t \rightarrow \infty$  with the same rate as the one of the heat equation. Moreover, after properly rescaled, the limit profile is the same as the one of the infinitesimal heat equation with diffusivity  $\mathfrak{q} = \frac{1}{2N} \int J(\xi)|\xi|^2 d\xi$  and the same initial condition.

These two facts can be seen as the two sides of a coin. They are both a consequence of the fact that for  $v \in C_0^\infty(\mathbb{R}^N)$  and  $J_\varepsilon(\xi) = \varepsilon^{-N} J(\varepsilon^{-1}\xi)$ ,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int J_\varepsilon(x-y)(v(y) - v(x)) dy = \mathfrak{q} \Delta v(x).$$

See, for instance, [27] for this discussion.

In this paper, after the study of the well posedness of problem (1.2), we address these kind of questions. We are interested in studying if and how the geometry of  $M$  influences the asymptotics of the nonlocal problem (1.2) both in the infinitesimal limit and in the large time behavior. In particular, in order to understand the influence of geometry at infinity we will concentrate on hyperbolic space. Note that (local) diffusions on hyperbolic space and on general noncompact manifolds have been considered by many authors (without being exhaustive, we cite [19, 24, 14, 31, 7]).

Our first result on the infinitesimal limit states that, for radially symmetric manifolds  $M$  the infinitesimal limit is the Heat-Beltrami problem (Theorem 3.1).

On the other hand, as can be expected, there is a big difference in the time asymptotics between compact and non-compact manifolds. Surprisingly, on the  $N$ -dimensional hyperbolic space, which is the model of an unbounded manifold with negative curvature, we find stunning differences between infinitesimal and large time asymptotics, as opposed to the Euclidean case.

Let us outline the results on this article.

In Section 2 we prove our first result that states that for every integrable initial datum there exists a unique solution in  $C([0, \infty); L^1(M)) \cap L^\infty(0, \infty; L^1(M))$ .

We also prove existence and uniqueness as well as a maximum principle for bounded initial data. Then we prove a comparison result for bounded sub- and supersolutions.

In order to study the infinitesimal limit we consider, in Section 3, a family of rescaled operators

$$L_\varepsilon u(x) := \frac{1}{\varepsilon^2} \int_M \varepsilon^{-N} J\left(\frac{xy}{\varepsilon}\right) (u(y) - u(x)) d\mu_y$$

with  $M$  a spherically symmetric  $C^2$  manifold and  $J$  with compact support, and we prove that for every  $u \in L^\infty(0, T; C_{loc}^{2+\alpha}(M))$  there holds that

$$L_\varepsilon u(x) \rightarrow \mathfrak{q} \Delta_M u(x)$$

locally uniformly in  $M \times [0, T]$  where  $\Delta_M$  is the Laplace-Beltrami operator on  $M$  and  $\mathfrak{q}$  is a constant that depends on  $J$ . When  $u \in L^\infty(0, T, C^{2+\alpha}(M))$  the convergence is uniform.

From this result we are able to prove that the solutions  $u^\varepsilon$  of the rescaled problems

$$\begin{cases} u_t^\varepsilon - L_\varepsilon u^\varepsilon = 0 & \text{in } M \times (0, T), \\ u^\varepsilon(x, 0) = u_0(x) & \text{in } M \end{cases}$$

converge uniformly in  $M \times [0, T]$  to the solution  $u$  of the Heat-Beltrami equation

$$\begin{cases} u_t - \Delta_M u = 0 & \text{in } M \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } M \end{cases}$$

if  $u \in L^\infty(0, T, C^{2+\alpha}(M))$ . This regularity is attained if, for instance,  $M \in C^3$  and  $u_0$  belongs to the closure of  $C^3(M)$  in the norm of  $C^{2+\alpha}(M)$  (see [25]).

We begin the study of the large time asymptotics in Section 4 by looking at the spectral properties of the operator  $L$  in the case of compact manifolds. We prove that there is a non-decreasing sequence of eigenvalues converging to one, the first one being 0. The normalized eigenfunctions form an orthonormal basis of  $L^2(M)$ . We deduce that every solution with initial datum in  $L^2(M)$  converges to its average exponentially fast in  $L^2(M)$ . For bounded initial data we prove this convergence in  $L^\infty(M)$ . We note at this point that in contrast with the usual heat operator, there is no regularizing effect. In particular, if the initial datum is not bounded the solution will not be bounded at any time. Nevertheless, the eigenfunctions are as smooth as the kernel  $J$  in the nonlocal operator.

Then, in Section 5 we turn our attention to the case of the  $N$ -dimensional hyperbolic space as an example of an unbounded, radially symmetric space. In this case, the infinitesimal limit corresponds to the Heat-Beltrami equation. But, using Fourier transform methods as in [10] for the Euclidean case, we find that, in contrast to the results in [10], the large time asymptotics for problem (1.2) are more related to a translated Heat-Beltrami equation. In fact, the final profile is the same as the one for the problem

$$\begin{cases} v_t = (a - 1)v + b \left( \Delta_{\mathbb{H}^N} v + \frac{(N-1)^2}{4} v \right), & x \in \mathbb{H}^N, t > 0, \\ v(x, 0) = u_0(x), & x \in \mathbb{H}^N. \end{cases}$$

Here  $a > 0$  is a weighted integral of the kernel  $J$  related to the radial Fourier transform and  $b > 0$  is also related to the Fourier transform of  $J$ .

Finally, there are several interesting features to be discussed. First, it might be more accurate to normalize the kernel  $J$  in the case of hyperbolic space in such a way that  $a = 1$  instead of the usual normalization (1.1). Then, the solutions will behave exactly as those of the translated problem

$$\begin{cases} v_t = b \left( \Delta_{\mathbb{H}^N} v + \frac{(N-1)^2}{4} v \right), & x \in \mathbb{H}^N, t > 0, \\ v(x, 0) = u_0(x), & x \in \mathbb{H}^N, \end{cases}$$

that may therefore be seen as a more natural model for diffusion on hyperbolic space than the Heat-Beltrami problem. In particular, in this case the solution decays as  $t^{-3/2}$  in any space dimension  $N \geq 2$  as opposed to the rate  $t^{-N/2}$  in Euclidean space, and the exponential decay of the solution of the Heat-Beltrami equation in hyperbolic space.

We include a discussion on these issues at the end of Section 5.

This model has already been proposed as a ‘‘natural’’ infinitesimal diffusion problem in hyperbolic space in previous articles (see, for instance, [1, 26]).

## 2. EXISTENCE, UNIQUENESS AND COMPARISON.

In this section we prove existence, uniqueness and comparison of solutions to the Cauchy problem (1.2). Since the proofs are entirely similar to the ones in  $\mathbb{R}^N$  we will only give the main ideas.

To begin with, let us notice that for every  $x \in M$  there holds that

$$\int_M J(s_{xy}) d\mu_y = 1.$$

In fact, since for every  $z \in M$ ,  $s_{xy} = s_{\tau_z(x)\tau_z(y)}$ . There holds that  $s_{xy} = s_{O\tau_x(y)}$ . Now, since  $d\mu$  is invariant under the change of variables  $z = \tau_x(y)$ , there holds that

$$\int_M J(s_{xy}) d\mu_y = \int_M J(s_{Oz}) d\mu_z = 1 \quad \text{by hypothesis.}$$

Observe that  $u$  is a solution to (1.2) if and only if

$$(2.1) \quad u(x, t) = e^{-t}u_0(x) + \int_0^t \int_M e^{-(t-r)} J(s_{xy}) u(y, r) d\mu_y dr.$$

This is a key point in the proof of existence and uniqueness. There holds,

**Theorem 2.1.** *Let  $u_0 \in L^1(M)$ . There exists a unique  $u \in C([0, \infty); L^1(M))$  solution to (1.2), and there holds that  $\|u(\cdot, t)\|_{L^1(M)} \leq \|u_0\|_{L^1(M)}$ .*

*If moreover,  $u_0 \in L^1(M) \cap L^\infty(M)$  there holds that  $\|u(\cdot, t)\|_{L^\infty(M)} \leq \|u_0\|_{L^\infty(M)}$ .*

*Proof.* It is enough to show that there exists  $t_0$  independent of  $u_0$  such that the operator

$$\mathcal{T}v(x, t) := e^{-t}u_0(x) + \int_0^t \int_M e^{-(t-r)} J(s_{xy}) v(y, r) d\mu_y dr$$

is a strict contraction in a closed, invariant subset of  $C([0, t_0]; L^1(M))$ . Since  $t_0$  is independent of  $u_0$  we can continue with the fixed point argument starting from  $t_0$ . In this way, we get a solution in  $C([0, \infty); L^1(M))$ .

The proof is very easy and completely similar to the one in  $\mathbb{R}^N$  (see [12]). In fact, since the set

$$\mathcal{K} = \{v \in C([0, t_0]; L^1(M)), \|v(\cdot, t)\|_{L^1(M)} \leq \|u_0\|_{L^1(M)}\}$$

is closed in the norm of  $C([0, t_0]; L^1(M))$ ,  $\mathcal{T}v(x, t) \in \mathcal{K}$  if  $v \in \mathcal{K}$  and  $\mathcal{T}$  is a strict contraction in  $\mathcal{K}$  if  $t_0$  is small depending only on  $\|u_0\|_{L^1(M)}$ , there holds that there exists a unique solution in  $\mathcal{K}$ .

Analogously, if  $u_0 \in L^1(M) \cap L^\infty(M)$  and we now let

$$\tilde{\mathcal{K}} = \{v \in \mathcal{K}, \|v(\cdot, t)\|_{L^\infty(M)} \leq \|u_0\|_{L^\infty(M)}\}$$

we again get a closed set invariant under the operator  $\mathcal{T}$ , and we get that the unique fixed point in  $\mathcal{K}$  belongs to  $\tilde{\mathcal{K}}$ . Hence,  $\|u(\cdot, t)\|_{L^\infty(M)} \leq \|u_0\|_{L^\infty(M)}$ .  $\square$

Now, we get a comparison result between continuous, bounded sub- and supersolutions. There holds,

**Proposition 2.1.** *Assume  $J \in L^1(M, (1 + s_{xO}^2) d\mu)$ . Let  $u, v \in C(M \times [0, T]) \cap L^\infty(M \times (0, T))$  be such that*

$$\begin{aligned} u_t - Lu &\leq v_t - Lv && \text{in } M \times (0, T], \\ u(x, 0) &\leq v(x, 0) && \text{in } M. \end{aligned}$$

*Then,  $u \leq v$ .*

*Proof.* We divide the proof into two cases.

First, assume  $M$  is a compact manifold.

Let  $w = v - u + \delta t$  and assume that  $w$  is negative somewhere in  $M \times [0, T]$ . Then, there exists  $(x_0, t_0) \in M \times (0, T]$  such that  $w(y, t) \geq w(x_0, t_0)$  for every  $(y, t) \in M \times (0, T]$ . Thus,

$$0 < w_t(x_0, t_0) - \int_M J(s_{xy})(w(y, t_0) - w(x_0, t_0)) d\mu_y \leq 0$$

which is a contradiction. Therefore,  $w \geq 0$ . Letting  $\delta \rightarrow 0$ , we get that  $u \leq v$ .

Now, assume that  $M$  is unbounded. Let

$$V(x, t) = 4\mathfrak{q}Nt + \left(e^t - \frac{1}{2}\right)s_{xO}^2$$

where  $\mathfrak{q} = \frac{1}{2N} \int_M J(s_{xO})s_{xO}^2 d\mu_x$ .

Then, since  $s_{yO}^2 \leq 2s_{xO}^2 + 2s_{xy}^2$  (this is a consequence of the triangular inequality),

$$s_{yO}^2 - s_{xO}^2 \leq s_{xO}^2 + 2s_{xy}^2.$$

Thus,

$$\int_M J(s_{xy})(s_{yO}^2 - s_{xO}^2) d\mu_y \leq s_{xO}^2 + 2 \int_M J(s_{xy})s_{xy}^2 d\mu_y = s_{xO}^2 + 4\mathfrak{q}N$$

because  $\int_M J(s_{xy})s_{xy}^2 d\mu_y = \int_M J(s_{zO})s_{zO}^2 d\mu_z = 2\mathfrak{q}N$ .

Let now  $\varepsilon, \delta > 0$  and for  $0 \leq t \leq \log \frac{3}{2} - \varepsilon$ , let

$$W(x, t) = w(x, t) + \delta V(x, t).$$

Then,

$$W_t - LW \geq \delta \left[ 4\mathfrak{q}N \left( \frac{3}{2} - e^t \right) + \frac{1}{2}s_{xO}^2 \right] > 0.$$

Moreover, there exists  $R_\delta$  such that, if  $s_{x0} \geq R_\delta$ ,  $t \geq 0$  there holds that  $W(x, t) > 0$ . Hence, if  $W$  is negative somewhere in  $M \times [0, \log \frac{3}{2} - \varepsilon]$ , it attains a minimum at some point  $(x_0, t_0)$  with  $t_0 > 0$  and, as in the previous case, we get a contradiction.

Therefore,  $W \geq 0$  for  $0 \leq t \leq \log \frac{3}{2} - \varepsilon$ .

Letting first  $\delta \rightarrow 0$  and then,  $\varepsilon \rightarrow 0$  we conclude that  $u \leq v$  for  $0 \leq t \leq \log \frac{3}{2}$ .

We can proceed in a similar way for  $\log \frac{3}{2} \leq t \leq 2 \log \frac{3}{2}$ , etc and we get the inequality  $u \leq v$  for  $0 \leq t \leq T$ .  $\square$

**Remark 2.1.** Observe that the comparison principle holds as long as the operator  $L$  can be written as

$$Lv(x) = c_0 \int_M J(s_{xy})(u(y) - u(x)) d\mu_y$$

independently of the value of  $c_0 > 0$ .

We also have existence and uniqueness of bounded solutions as well as a maximum principle, by assuming only that  $u_0 \in L^\infty(M)$ .

**Theorem 2.2.** *Let  $u_0 \in L^\infty(M)$ . There exists a unique solution to (1.2) in  $C([0, \infty), L^\infty(M))$ . Moreover, this solution  $u$  satisfies that  $\|u\|_{L^\infty(M \times (0, \infty))} \leq \|u_0\|_{L^\infty(M)}$ .*

*Proof.* It is easy to see that

$$\widehat{\mathcal{K}} = \{v \in C([0, t_0], L^\infty(M)), \|u\|_{L^\infty(M \times (0, \infty))} \leq \|u_0\|_{L^\infty(M)}\}$$

is closed under the map  $\mathcal{T}$ , and this map is a strict contraction in  $\widehat{\mathcal{K}}$  if  $t_0$  is small (independent of  $u_0$ ). Therefore,  $\mathcal{T}$  has a unique fixed point. Iterating this argument, we get a unique global solution.  $\square$

**Remark 2.2.** Observe that when  $u_0 \in C(M) \cap L^\infty(M)$  and  $u \in C([0, \infty), L^\infty(M))$  is the unique solution with initial datum  $u_0$ , there holds that  $u \in C(M \times [0, T])$ . This can be seen from (2.1), for instance.

The same result is true if  $u_0 \in C(M) \cap L^1(M)$  and  $J \in L^1 \cap L^\infty$ .

### 3. THE LAPLACE BELTRAMI OPERATOR AS THE INFINITESIMAL LIMIT

In this section we assume that  $J$  has compact support contained in  $[0, 1]$ . Let  $M$  be a spherically symmetric  $C^2$  manifold which has the property that at any point in  $M$ , denoted by  $O$  there exists a small ball  $B_R$  centered at  $O$  and polar coordinates  $(r, \theta)$  such that the Riemannian metric in  $B_R \setminus O$  is given by

$$ds^2 = dr^2 + \psi^2(r)d\theta^2,$$

where  $d\theta^2$  is the standard metric on  $\mathbb{S}^{N-1}$ ,  $\psi \in C^2$ ,  $\psi(0) = 0$  and  $\psi'(0) \neq 0$ , see [19]. Without loss of generality we may assume that  $\psi'(0) = 1$ . Here we have made use of the assumption that there exists for every point an isometry  $\tau_x$  such that  $\tau_x(x) = O$ .

Notice that if  $\psi = r$  then  $M = \mathbb{R}^N$ , if  $\psi = \sin r$  then  $M = \mathbb{S}^N$  and if  $\psi = \sinh r$  then  $M = \mathbb{H}^N$ .

We consider the measure in  $M$  such that, in these coordinates, the volume element assumes the form

$$d\mu = \psi^{N-1}(r)drd\theta,$$

with  $d\theta$  the usual area element of the  $(N-1)$ -dimensional sphere, and the Laplace Beltrami operator on  $M$  is

$$\Delta_M = \partial_{rr} + (N-1)\frac{\psi'}{\psi}\partial_r + \frac{1}{\psi^2}\Delta_{\mathbb{S}^{N-1}}$$

where  $\Delta_{\mathbb{S}^{N-1}}$  is the Laplace Beltrami operator on  $\mathbb{S}^{N-1}$ . Choose  $R$  so small that  $B_R \subset M \setminus \{\text{cut locus of } O\}$ . It is easy to see that in this case the unique geodesic from any point  $y \in B_R$  to the center  $x$  of the ball  $B_R$  is a straight line and that the geodesic ball  $B_r$  centered at  $x$  is  $\{(s, \theta) : 0 < s < r, \theta \in \mathbb{S}^{N-1}\}$ .

In this section we consider a family of nonlocal diffusion operators defined on  $M$  which are obtained from  $L$  by rescaling. Namely,

$$L_\varepsilon u(x) = \frac{1}{\varepsilon^2} \int_M \varepsilon^{-N} J\left(\frac{xy}{\varepsilon}\right) (u(y) - u(x)) d\mu_y.$$

We prove that, when  $u \in C_{loc}^{2+\alpha}(M)$ , there holds that

$$(3.1) \quad L_\varepsilon u(x) \rightarrow \mathfrak{q}\Delta_M u(x) \quad \text{as } \varepsilon \rightarrow 0$$

locally uniformly in  $M$  with  $\Delta_M$  the Laplace Beltrami operator on  $M$ . If  $u \in C^{2+\alpha}(M)$ , the convergence is uniform and, moreover

$$(3.2) \quad |L_\varepsilon u(x) - \mathfrak{q}\Delta_M u(x)| \leq C\|u\|_{C^{2+\alpha}(M)} \varepsilon^\alpha \quad \forall x \in M.$$

**Theorem 3.1.** *Let  $M$  and  $L_\varepsilon$  be as above. Let  $\mathfrak{q} = \frac{1}{2N} \int_{\mathbb{R}^N} J(|z|) |z|^2 dz$ . Then, (3.1) holds locally uniformly in  $M$ . If  $u \in C^{2+\alpha}(M)$  globally, the convergence is uniform in  $M$  and (3.2) holds.*

*Proof.* Let us now introduce polar coordinates in a small ball  $B_R$  centered at  $x$ . In this case  $s_{xy} = \rho$  and thus, taking into account that the support of  $J$  is contained in  $[0, 1]$ , we obtain for  $\varepsilon$  small enough

$$L_\varepsilon u(x) = \varepsilon^{-(N+2)} \int_0^\varepsilon J\left(\frac{\rho}{\varepsilon}\right) \int_{\mathbb{S}^{N-1}} (u(\rho, \theta) - u(x)) \psi^{N-1}(\rho) d\theta d\rho.$$

Since  $u \in C_{loc}^{2+\alpha}(M)$  we have,

$$\begin{aligned} u(\rho, \theta) - u(x) &= u_\rho(\rho, \theta)\rho - \frac{1}{2}u_{\rho\rho}(\rho, \theta)\rho^2 + O(\rho^{2+\alpha}) \\ &= u_\rho(\rho, \theta)\rho - \frac{1}{2}\Delta_M u(\rho, \theta)\rho^2 + \frac{N-1}{2} \frac{\psi'(\rho)\rho}{\psi(\rho)} \rho u_\rho(\rho, \theta) \\ &\quad + \frac{1}{2} \frac{\rho^2}{\psi^2(\rho)} \Delta_{\mathbb{S}^{N-1}} u(\rho, \theta) + O(\rho^{2+\alpha}). \end{aligned}$$

Observe that  $u_\rho(\rho, \theta)$  is the derivative in the direction of the outer normal to  $B_\rho$ . Therefore, by applying the divergence theorem on manifolds we get,

$$\begin{aligned} \int_{\mathbb{S}^{N-1}} (u(\rho, \theta) - u(x)) \psi^{N-1}(\rho) d\theta &= \rho \left(1 + \frac{N-1}{2} \frac{\psi'(\rho)\rho}{\psi(\rho)}\right) \int_{B_\rho(x)} \Delta_M u(y) d\mu_y \\ &\quad - \frac{1}{2} \int_{\mathbb{S}^{N-1}} \Delta_M u(\rho, \theta) \psi^{N-1}(\rho) \rho^2 d\theta - \frac{\rho^2}{\psi^2(\rho)} \int_{\mathbb{S}^{N-1}} \Delta_{\mathbb{S}^{N-1}} u(\rho, \theta) \psi^{N-1}(\rho) d\theta + O(\rho^{2+\alpha}) \psi^{N-1}(\rho). \end{aligned}$$

Now, taking into account that  $\int_{\mathbb{S}^{N-1}} \Delta_{\mathbb{S}^{N-1}} u(\rho, \theta) d\theta = 0$ ,  $\frac{\psi'(\rho)\rho}{\psi(\rho)} = 1 + O(\rho)$ ,  $\psi^{N-1}(\rho) = \rho^{N-1} + O(\rho^N)$ , and  $|B_\rho| = \omega_{N-1}(\frac{\rho^N}{N} + O(\rho^{N+1}))$  with  $\omega_{N-1}$  the usual measure of the sphere  $\mathbb{S}^{N-1}$  we get,

$$\begin{aligned} &\int_{\mathbb{S}^{N-1}} (u(\rho, \theta) - u(x)) \psi^{N-1}(\rho) d\theta \\ &= \rho \frac{N+1}{2} \int_{B_\rho(x)} \Delta_M u(y) d\mu_y - \frac{\rho^{N+1}}{2} \int_{\mathbb{S}^{N-1}} \Delta_M u(\rho, \theta) d\theta + O(\rho^{N+1+\alpha}) \\ &= \rho^{N+1} \left(\frac{N+1}{2N} - \frac{1}{2}\right) \omega_{N-1} \Delta_M u(x) + O(\rho^{N+1+\alpha}) \end{aligned}$$

Therefore,

$$L_\varepsilon u(x) = \frac{\omega_{N-1}}{2N} \left[ \int_0^\varepsilon J\left(\frac{\rho}{\varepsilon}\right) \left(\frac{\rho}{\varepsilon}\right)^2 \left(\frac{\rho}{\varepsilon}\right)^{N-1} \frac{d\rho}{\varepsilon} \right] \Delta_M u(x) + O(\varepsilon^\alpha) = \mathfrak{q} \Delta_M u(x) + O(\varepsilon^\alpha).$$

And the theorem is proved.

Observe that the error term depends only on  $\|\psi\|_{C^2(B)}$  and  $\|u\|_{C^{2+\alpha}(B)}$  where  $B$  is a small ball around  $x$ . And, if  $u \in C^{2+\alpha}(M)$ , (3.2) holds with a constant depending only on  $\psi$ .  $\square$

Now we have the elements to prove the convergence of the solution of the  $\varepsilon$ -problem to the solution of the Heat-Beltrami equation. There holds,

**Theorem 3.2.** *Let  $u \in C^{2+\alpha, 1+\alpha/2}(M \times [0, T])$  be the solution to*

$$\begin{cases} u_t - \mathfrak{q}\Delta_M u = 0 & \text{in } M \times (0, T], \\ u(x, 0) = u_0(x) & \text{in } M. \end{cases}$$

*Let  $u^\varepsilon$  be the solution to*

$$\begin{cases} u_t - L_\varepsilon u = 0 & \text{in } M \times (0, T], \\ u(x, 0) = u_0(x) & \text{in } M. \end{cases}$$

*Then,  $u^\varepsilon \rightarrow u$  uniformly in  $M \times [0, T]$ .*

*Proof.* Let  $w^\varepsilon = u - u^\varepsilon$ . Then,  $w^\varepsilon$  is a continuous solution to

$$\begin{cases} w_t^\varepsilon - L_\varepsilon w^\varepsilon = \mathfrak{q}\Delta_M u - L_\varepsilon u := F_\varepsilon & \text{in } M \times (0, T), \\ w^\varepsilon(x, 0) = 0 & \text{in } M. \end{cases}$$

Recall that we have  $F_\varepsilon \rightarrow 0$  uniformly in  $M \times (0, T)$ .

Let  $z^\varepsilon(x, t) = t \|F_\varepsilon\|_{L^\infty(M \times (0, T))}$ . Then,  $z^\varepsilon$  is the unique bounded solution to

$$\begin{cases} z_t^\varepsilon - L_\varepsilon z^\varepsilon = \|F_\varepsilon\|_{L^\infty(M \times (0, T))} & \text{in } M \times (0, T), \\ z^\varepsilon(x, 0) = 0 & \text{in } M. \end{cases}$$

So, by the comparison principle (Proposition 2.1 and Remark 2.1) we have

$$-z^\varepsilon \leq w^\varepsilon \leq z^\varepsilon.$$

Since  $z^\varepsilon \rightarrow 0$  uniformly in  $M \times (0, T)$ , the theorem is proved.  $\square$

**Remark 3.1.** Let  $M$  be a spherically symmetric  $C^3$  manifold and  $u_0$  in the closure of  $C^3(M)$  in the norm of  $C^{2+\alpha}(M)$ . Let  $u$  be the unique bounded solution of the Heat-Beltrami equation in  $M \times (0, T)$  such that  $u \in C([0, T]; L^\infty(M))$ . Then,  $u \in C([0, T]; C^{2+\alpha}(M))$ . This is a consequence of the uniqueness of bounded solution and the existence of solution in  $C([0, T]; C^{2+\alpha}(M))$  (see [25], Theorem 3.7).

#### 4. SPECTRAL PROPERTIES IN COMPACT MANIFOLDS.

In this section we study the spectrum of  $L$  when  $M$  is compact and connected. We will assume throughout the section that  $J(s)$  is positive for  $0 \leq s < s_0$  for a certain  $s_0 > 0$  and  $J$  is Lipschitz in  $M$ . From the spectral properties we deduce the large time asymptotics of the solutions. We begin by assuming that  $u_0 \in L^2(M)$ . Then, we consider the case of bounded initial data in order to get uniform decay estimates. We recall that  $u$  is a solution to the integral equation (2.1) from which it is clear that if  $u_0 \in L^1(M)$  is not bounded,  $u(\cdot, t)$  will not be bounded for any  $t > 0$ .

**Theorem 4.1.** *Let us consider  $L$  as an operator in  $L^2(M)$ . There exists a sequence  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \nearrow 1$  of eigenvalues of  $L$  and corresponding eigenfunctions  $\{\varphi_k\}$  that form an orthonormal basis in  $L^2(M)$ .*

*Proof.* As already observed in [17] for the case of  $\mathbb{R}^N$ , we can write  $L = L_0 - I$  where  $I$  is the identity operator and

$$L_0 u(x) = \int_M J(s_{xy}) u(y) d\mu_y.$$

We then deduce that  $\lambda$  is in the spectrum of  $-L$  if and only if  $\gamma = 1 - \lambda$  is in the spectrum of  $L_0$ .



It is easy to see that  $L_0$  is a positive, symmetric operator. Moreover, since  $J$  is smooth there holds that  $L_0 u \in C(M)$  for every  $u \in L^2(M)$  and,

$$\|L_0 u\|_{L^\infty(M)} \leq \|J(s_{x0})\|_{L^2(M)} \|u\|_{L^2(M)}, \quad \|L_0 u(x) - L_0 u(\bar{x})\|_{L^\infty(M)} \leq C_{J,M} \|u\|_{L^2(M)} s_{x\bar{x}}.$$

Thus,  $L_0$  is a compact operator from  $L^2(M)$  to  $C(M)$  and therefore, also in  $L^2(M)$ . Moreover, its spectrum consists of a nonincreasing sequence of positive eigenvalues converging to 0. We conclude that the spectrum of  $-L$  consists of a nondecreasing sequence of eigenvalues converging to 1. It is easy to see that all the eigenvalues are nonnegative. This is, if  $-L\varphi = \lambda\varphi$  and  $\varphi \neq 0$ , there holds that  $\lambda \geq 0$ . In fact, by multiplying the equation by  $\varphi$  and applying Fubini's theorem we get,

$$\begin{aligned} \lambda \int_M \varphi^2 d\mu &= - \int_M \int_M J(s_{xy})(\varphi(y) - \varphi(x))\varphi(x) d\mu_y d\mu_x \\ &= \frac{1}{2} \int_M \int_M J(s_{xy})(\varphi(y) - \varphi(x))^2 d\mu_x d\mu_y \geq 0. \end{aligned}$$

Then we must have  $\lambda \geq 0$ .

Moreover, since the constants are solutions to  $Lu = 0$ , there holds that 0 is an eigenvalue of  $L$ . Since  $M$  is connected, this eigenvalue is simple. In fact, only the constants are solutions of the homogeneous equation: if  $L\varphi = 0$  then,  $\varphi = L_0\varphi$  is a smooth function. Since  $M$  is compact, there exists  $x_0 \in M$  such that  $\varphi(x_0) = \max \varphi$ . Let  $A$  be the set of points where  $\varphi(x) = \varphi(x_0)$ . Then,  $A$  is a closed, nonempty set. Let us see that it is open. In fact, if  $\bar{x} \in A$ ,

$$\varphi(\bar{x}) = \int_M J(s_{\bar{x}y})\varphi(y) d\mu_y \leq \varphi(\bar{x}).$$

As a consequence, there is equality and  $\varphi(y)$  must be identically equal to  $\varphi(\bar{x})$  in the support of  $J(s_{\bar{x}y})$  that contains a neighborhood of  $\bar{x}$  by assumption.

Moreover, by the general spectral theory of positive, symmetric, compact operators in a Hilbert space, there is a set of eigenfunctions of  $L_0$  associated to the sequence of eigenvalues  $\gamma_k$  that is an orthonormal basis of  $L^2(M)$ . Since eigenfunctions of  $L_0$  associated to  $\gamma_k \searrow 0$  are eigenfunctions of  $-L$  associated to  $\lambda_k = 1 - \gamma_k \nearrow 1$ , the theorem is proved.  $\square$

As a consequence, we have the following result:

**Corollary 4.1.** *Let  $u_0 \in L^2(M)$ . Let  $u$  be the solution to (1.2). Then,*

$$\|u(\cdot, t) - \langle u_0 \rangle\|_{L^2(M)} \leq e^{-\lambda_1 t} \|u_0\|_{L^2(M)}$$

with  $\langle u_0 \rangle = \int_M u_0(x) d\mu_x$ , the mean value of  $u_0$  on  $M$  and  $\lambda_1 > 0$  the first nonzero eigenvalue of  $-L$ .

*Proof.* By separation of variables we have that

$$u(x, t) = \sum_{k=0}^{\infty} c_k e^{-\lambda_k t} \varphi_k(x)$$

with the sum converging in  $L^2(M)$  uniformly in  $t$ , and  $c_k = \int_M u_0(x) \varphi_k(x) d\mu_x$ . Since there holds that  $c_0 \varphi_0(x) = \langle u_0 \rangle$  and  $\lambda_k \geq \lambda_1 > 0$  for  $k \geq 1$ , we have the result.  $\square$

Now, if moreover  $u_0 \in L^\infty(M)$  we get the same decay in  $L^\infty$  norm. In fact,

**Corollary 4.2.** *Let  $u_0 \in L^\infty(M)$ . Let  $u$  be the solution to (1.2). Then,*

$$\|u(\cdot, t) - \langle u_0 \rangle\|_{L^\infty(M)} \leq C e^{-\lambda_1 t} \|u_0\|_{L^\infty(M)}$$

with  $\langle u_0 \rangle = \int_M u_0(x) d\mu_x$ , the mean value of  $u_0$  on  $M$  and  $\lambda_1 > 0$  the first nonzero eigenvalue of  $-L$ .

*Proof.* Recall that

$$u(x, t) = e^{-t} u_0(x) + \int_0^t \int_M e^{-(t-r)} J(s_{xy}) u(y, r) d\mu_y dr.$$

Using that  $\int_0^t e^{-(t-r)} dr = 1 - e^{-t}$ , we have

$$\begin{aligned} u(x, t) - \langle u_0 \rangle &= e^{-t} (u_0(x) - \langle u_0 \rangle) + \int_0^t e^{-(t-r)} \int_M J(s_{xy}) (u(y, r) - \langle u_0 \rangle) d\mu_y dr \\ &\leq C e^{-t} \|u_0\|_\infty + C_J \|u_0\|_2 \int_0^t e^{-(t-r)} e^{-\lambda_1 r} dr \\ &= C e^{-t} \|u_0\|_\infty + C_J \|u_0\|_2 \frac{e^{-\lambda_1 t} - e^{-t}}{1 - \lambda_1}. \end{aligned}$$

Since  $1 - \lambda_1 = \gamma_1 > 0$ , we have that

$$|u(x, t) - \langle u_0 \rangle| \leq \frac{\bar{C}}{1 - \lambda_1} (\|u_0\|_\infty + \|u_0\|_2) e^{-\lambda_1 t},$$

and we immediately obtain the desired conclusion.  $\square$

## 5. TIME ASYMPTOTICS ON HYPERBOLIC SPACE

In this section we study the large time asymptotics of the solution of (1.2) in the case of the hyperbolic space  $\mathbb{H}^N$ ,  $N \geq 2$ .

For the sake of clarity we divide the section into several subsections where we recall the definition and properties of the Fourier transform on hyperbolic space and the study of the solution of translations of the Heat-Beltrami operator, among other results and ideas that are used in the last subsections in order to prove the main result of this section, Theorem 5.1.

**5.1. Preliminaries on hyperbolic space and its Fourier transform.** Hyperbolic space  $\mathbb{H}^N$  may be defined as the upper branch of a hyperboloid in  $\mathbb{R}^{N+1}$  with the metric induced by the Lorentzian metric in  $\mathbb{R}^{N+1}$  given by  $-dx_0^2 + dx_1^2 + \dots + dx_N^2$ , i.e.,  $\mathbb{H}^N = \{(x_0, \dots, x_N) \in \mathbb{R}^{N+1} : x_0^2 - x_1^2 - \dots - x_N^2 = 1, x_0 > 0\}$ , which in polar coordinates may be parameterized as

$$\mathbb{H}^N = \{x \in \mathbb{R}^{N+1} : x = (\cosh r, \sinh r \theta), r \geq 0, \theta \in \mathbb{S}^{N-1}\},$$

with the metric  $g_{\mathbb{H}^N} = dr^2 + \sinh^2 r d\theta^2$ , where  $d\theta^2$  is the canonical metric on  $\mathbb{S}^{N-1}$ . Under these definitions the Laplace-Beltrami operator is given by

$$\Delta_{\mathbb{H}^N} = \partial_{rr} + (N-1) \frac{\cosh r}{\sinh r} \partial_r + \frac{1}{\sinh^2 r} \Delta_{\mathbb{S}^{N-1}},$$

and the volume element is

$$d\mu_x = \sinh^{N-1} r dr d\theta.$$

We denote by  $[\cdot, \cdot]$  the internal product induced by the Lorentzian metric

$$[x, x'] = x_0 x'_0 - x_1 x'_1 - \dots - x_N x'_N.$$

The hyperbolic distance between two arbitrary points is given by

$$s_{xx'} = d(x, x') = \cosh^{-1}([x, x']),$$

and in the particular case that  $x = (\cosh r, \sinh r \omega)$ ,  $x' = O$ ,

$$s_{xO} = d(x, O) = r.$$

The unit sphere  $\mathbb{S}^{N-1}$  is identified with the subset  $\{x \in \mathbb{R}^{N+1} : [x, x] = 0, x_0 = 1\}$  via the map  $b(\omega) = (1, \omega)$  for  $\omega \in \mathbb{S}^{N-1}$ .

Finally, note that hyperbolic space may be written as a symmetric space of rank one as the quotient  $\mathbb{H}^N \approx \frac{SO(1, N)}{SO(N)}$ .

Now we start by reviewing some basic facts about the Fourier transform on hyperbolic space, which is a particular case of the Helgason-Fourier transform on symmetric spaces. Some standard references are [8, 18, 21, 28, 29]. First, the generalized eigenfunctions of the Laplace-Beltrami operator may be written as

$$h_{\lambda, \omega}(x) = [x, b(\omega)]^{i\lambda - \frac{N-1}{2}}, \quad x \in \mathbb{H}^N,$$

where  $\lambda \in \mathbb{R}$  and  $\omega \in \mathbb{S}^{N-1}$ . These satisfy

$$\Delta_{\mathbb{H}^N} h_{\lambda, \omega} = -\left(\lambda^2 + \frac{(N-1)^2}{4}\right) h_{\lambda, \omega}.$$

In analogy to the Euclidean space, the Fourier transform on  $\mathbb{H}^N$  is defined by

$$\hat{u}(\lambda, \omega) = \int_{\mathbb{H}^N} u(x) h_{\lambda, \omega}(x) d\mu_x,$$

for  $\lambda \in \mathbb{R}$ ,  $\omega \in \mathbb{S}^{N-1}$ . Moreover, the following inversion formula holds:

$$(5.1) \quad u(x) = \int_{-\infty}^{\infty} \int_{\mathbb{S}^{N-1}} \bar{h}_{\lambda, \omega}(x) \hat{u}(\lambda, \omega) \frac{d\omega d\lambda}{|c(\lambda)|^2},$$

where  $c(\lambda)$  is the Harish-Chandra coefficient:

$$\frac{1}{|c(\lambda)|^2} = \frac{1}{2(2\pi)^N} \frac{|\Gamma(i\lambda + \frac{N-1}{2})|^2}{|\Gamma(i\lambda)|^2}.$$

There is also a Plancherel formula:

$$\int_{\mathbb{H}^N} |u(x)|^2 d\mu_x = \int_{\mathbb{R} \times \mathbb{S}^{N-1}} |\hat{u}(\lambda, \omega)|^2 \frac{d\omega d\lambda}{|c(\lambda)|^2},$$

which implies that the Fourier transform extends to an isometry of  $L^2(\mathbb{H}^N)$  onto  $L^2(\mathbb{R}_+ \times \mathbb{S}^{N-1}, |c(\lambda)|^{-2} d\lambda d\omega)$ .

If  $u$  is a radial function, then  $\hat{u}$  is also radial, and the above formulas simplify. In this setting, it is customary to normalize the measure of  $\mathbb{S}^{N-1}$  to one in order not to account for multiplicative constants. Thus one defines the spherical Fourier transform as

$$\hat{u}(\lambda) = \int_{\mathbb{H}^N} u(x) \Phi_{-\lambda}(x) d\mu_x$$

where

$$\Phi_{\lambda}(x) = \int_{\mathbb{S}^{N-1}} h_{-\lambda, \omega}(x) d\omega$$

is known as the elementary spherical function. In addition, (5.1) reduces to

$$u(x) = \int_{-\infty}^{\infty} \hat{u}(\lambda) \Phi_{\lambda}(x) \frac{d\lambda}{|c(\lambda)|^2}.$$

Using polar coordinates on  $\mathbb{H}^N$  (with some abuse in notation since  $\Phi_\lambda(x)$  is radial) we have

$$\Phi_\lambda(r) = c_N \int_0^\pi (\cosh r - \sinh r \cos \theta)^{i\lambda - \frac{N-1}{2}} (\sin \theta)^{N-2} d\theta.$$

Notice that

$$\Phi_{-\lambda}(r) = \Phi_\lambda(r) = \Phi_\lambda(-r),$$

and that we have normalized  $\Phi_\lambda(0) = 1$ . As a function of  $\lambda$ ,  $\Phi_\lambda$  is an even entire function on the whole complex plane, and in particular, we have the expansion:

$$\Phi_\lambda(r) = \Phi_0(r) + \partial_{\lambda\lambda}|_{\lambda=0} \Phi_\lambda(r) \frac{\lambda^2}{2} + o(\lambda^2).$$

Another integral formula is

$$\Phi_\lambda(r) = \frac{c_N}{\sinh^{N-2} r} \int_{-r}^r e^{i\lambda s} (\cosh r - \cosh s)^{\frac{N-1}{2}-1} ds,$$

from where it is easy to estimate

$$(5.2) \quad |\Phi_\lambda(r)| \leq |\Phi_0(r)| \leq C e^{-\frac{N-1}{2}r} (1+r).$$

For large  $|\lambda|$  the estimate may be improved to (see Lemma 2.2 in [26])

$$|\Phi_\lambda(r)| \leq \frac{C}{|\lambda|^{\frac{N-1}{2}}} e^{-\frac{N-1}{2}r} \quad \text{when } |\lambda| \rightarrow \infty.$$

Another result in the case of radially symmetric functions is the following (see [8], Thm. 3.3.1 (iii) or [22] where this result is actually proven): if  $u \in C_0^\infty(\mathbb{H}^N)$  is radially symmetric, there holds that  $\hat{u}(\lambda)$  can be extended to  $\mathbb{C}$  as an entire function that decays at infinity faster than any negative power of  $|\lambda|$ .

Finally, we define the convolution operator as

$$u * v(x) = \int_{\mathbb{H}^N} u(x') v(\tau_x^{-1} x') d\mu_{x'},$$

where  $\tau_x : \mathbb{H}^N \rightarrow \mathbb{H}^N$  is an isometry that takes  $x$  into  $O$ . If  $v$  is a radial function, then the convolution may be written as

$$u * v(x) = \int_{\mathbb{H}^N} u(x') v(s_{xx'}) d\mu_{x'},$$

and we have the property

$$(5.3) \quad \widehat{u * v} = \hat{u} \hat{v},$$

in analogy to the usual Fourier transform.

It is also interesting to observe that

$$\widehat{\Delta_{\mathbb{H}^N} u} = - \left( \lambda^2 + \frac{(N-1)^2}{4} \right) \hat{u}.$$

On hyperbolic space there is a well developed theory of Fourier multipliers. In  $L^2$  spaces everything may be written out explicitly. For instance, let  $m(\lambda)$  be a multiplier in Fourier variables.

A function  $\hat{u}(\lambda, \omega) = \hat{m}(\lambda)u_0(\lambda, \omega)$ , by the inversion formula for the Fourier transform (5.1) and expression (5.5), may be written as

$$(5.4) \quad \begin{aligned} u(x) &= \int_{-\infty}^{\infty} \int_{\mathbb{S}^{N-1}} m(\lambda) \hat{u}_0(\lambda, \omega) \bar{h}_{\lambda, \omega}(x) \frac{d\omega d\lambda}{|c(\lambda)|^2} \\ &= \int_{-\infty}^{\infty} \int_{\mathbb{H}^N} m(\lambda) u_0(x') k_\lambda(x, x') d\mu_{x'} d\lambda, \end{aligned}$$

where we have denoted

$$(5.5) \quad k_\lambda(x, x') = \frac{1}{|c(\lambda)|^2} \int_{\mathbb{S}^{N-1}} \bar{h}_{\lambda, \omega}(x) h_{\lambda, \omega}(x') d\omega.$$

It is known that  $k_\lambda$  is invariant under isometries, i.e.,

$$k_\lambda(x, x') = k_\lambda(\tau x, \tau x'),$$

for all  $\tau \in SO(1, N)$ , and in particular,

$$k_\lambda(x, x') = k_\lambda(s_{xx'}),$$

so many times we will simply write  $k_\lambda(\rho)$  for  $\rho = s_{xx'}$ . We recall the following formulas for  $k_\lambda$  (see, for instance, [2], which refers to [18]):

**Lemma 5.1.** *For  $N \geq 3$  odd,*

$$(5.6) \quad k_\lambda(\rho) = c_N \left( \frac{\partial_\rho}{\sinh \rho} \right)^{\frac{N-1}{2}} (\cos \lambda \rho),$$

and for  $N \geq 2$  even,

$$(5.7) \quad k_\lambda(\rho) = c_N \int_\rho^\infty \frac{\sinh s}{\sqrt{\cosh s - \cosh \rho}} \left( \frac{\partial_s}{\sinh s} \right)^{\frac{N}{2}} (\cos \lambda s) ds.$$

Before we state our results on hyperbolic space we prove some preliminary statements on the asymptotic behavior of the kernel  $k_\lambda$ :

**Lemma 5.2.** *For each  $m = 1, 2, \dots$ , the derivative*

$$\left( \frac{\partial_\rho}{\sinh \rho} \right)^m \cos(\lambda \rho)$$

is a finite linear combination of terms of the form

$$f(\rho^2)g(\lambda, \rho)$$

where  $f(z)$  is an analytic function on  $|z| \leq \pi^2$ , and  $g$  can be described in two different ways:

i. First,

$$g(\lambda, \rho) = \lambda^n g_1((\lambda \rho)^2)$$

where  $g_1(z)$  is an analytic function on  $\mathbb{C}$ , and  $n$  is some integer  $2 \leq n \leq 2m$ .

ii. Second,  $g$  can be written as a finite linear combination of terms of the form

$$\lambda^n \frac{\sin(\lambda \rho)}{(\lambda \rho)^l} \quad \text{or} \quad \lambda^n \frac{\cos(\lambda \rho)}{(\lambda \rho)^l},$$

for some nonnegative integers  $n, l$ , with  $2 \leq n \leq 2m$ .

*Proof.* By induction. For the initial step  $m = 1$ , we calculate

$$\left(\frac{\partial_\rho}{\sinh \rho}\right) \cos(\lambda\rho) = \lambda^2 \left(\frac{\rho}{\sinh \rho}\right) \left(\frac{-\sin(\lambda\rho)}{\lambda\rho}\right),$$

which is of the specified form. Then, for the step  $m + 1$ , first calculate

$$\frac{\partial_\rho}{\sinh \rho} \{f(\rho^2)g_1((\lambda\rho)^2)\} = \frac{2f'(\rho^2)\rho}{\sinh \rho} \cdot g((\lambda\rho)^2) + \frac{2f(\rho^2)\rho}{\sinh \rho} \cdot \lambda^2 g'((\lambda\rho)^2),$$

which has the desired form *i*. On the other hand, for *ii*. just notice that

$$\frac{\partial_\rho}{\sinh \rho} \left\{ f(\rho^2) \frac{\sin(\lambda\rho)}{(\lambda\rho)^l} \right\} = \frac{2f'(\rho^2)\rho}{\sinh \rho} \cdot \frac{\sin(\lambda\rho)}{(\lambda\rho)^l} + \frac{f(\rho^2)\rho}{\sinh \rho} \cdot \lambda^2 \frac{\cos(\lambda\rho)}{(\lambda\rho)^{l+1}} - l \frac{f(\rho^2)\rho}{\sinh \rho} \cdot \lambda^2 \frac{\sin(\lambda\rho)}{(\lambda\rho)^{l+2}},$$

and a similar calculation holds when  $\sin \lambda\rho$  is replaced by  $\cos \lambda\rho$ .

This concludes the proof of the Lemma.  $\square$

**Lemma 5.3.** For a function  $u = u(\rho)$ , and  $m = 1, 2, \dots$ ,

$$\left(\frac{\partial_\rho}{\sinh \rho}\right)^m u = \sum_{j=1}^m h_j^m(\rho) \partial_\rho^{(j)} u,$$

such that, for  $\rho \geq 1$ , we have the bound

$$|h_j^m(\rho)| \leq C_j \frac{1}{(\sinh \rho)^m}, \quad j = 1, \dots, m.$$

*Proof.* By induction in  $m$  we see that  $h_n^m(\rho)$  is a finite linear combination of terms of the form

$$\frac{(\cosh \rho)^j}{(\sinh \rho)^{j+m}}.$$

So, the result follows.  $\square$

**Corollary 5.1.** For every  $m = 1, 2, \dots$  we have the bound (for all  $\lambda \in \mathbb{R}$ )

$$(5.8) \quad \left| \left(\frac{\partial_\rho}{\sinh \rho}\right)^m \cos(\lambda\rho) \right| \leq \begin{cases} C \frac{\rho}{(\sinh \rho)^m} \sum_{n=2}^m \lambda^n, & \rho \geq 1, \\ C \sum_{n=2}^{2m} \lambda^n, & 0 < \rho \leq 1. \end{cases}$$

*Proof.* The estimate near  $\rho = 0$  follows from Lemma 5.2, using expansions *i* or *ii*. according to the value of  $\lambda\rho$ . On the other hand, for the bound for large  $\rho$  we apply Lemma 5.3 to get

$$\left| \left(\frac{\partial_\rho}{\sinh \rho}\right)^m \cos(\lambda\rho) \right| \leq Ch_1(\rho) |\partial_\rho \cos \lambda\rho| + \sum_{n=2}^m h_n(\rho) \lambda^n$$

with  $h_n(\rho) \leq C(\sinh \rho)^{-m}$  for every  $n \in \mathbb{N}$ . Moreover,

$$\partial_\rho \cos(\lambda\rho) = -\lambda \sin \lambda\rho = -\lambda^2 \rho \frac{\sin \lambda\rho}{\lambda\rho}$$

and the results follows.  $\square$

**Remark 5.1.** Observe that we also have

$$\left| \left(\frac{\partial_\rho}{\sinh \rho}\right)^m \cos(\lambda\rho) \right| \leq \frac{1}{(\sinh \rho)^m} \sum_{n=1}^m \lambda^n.$$

**Lemma 5.4.** *Let  $N \geq 2$  be an even integer. Assume that the function  $h$  satisfies*

$$(5.9) \quad |h(s)| \leq \begin{cases} \sinh^{-\frac{N}{2}} s, & s \geq 1, \\ 1, & 0 < s \leq 1. \end{cases}$$

Then the integral

$$I(\rho) := \int_{\rho}^{\infty} \frac{\sinh s}{\sqrt{\cosh s - \cosh \rho}} h(s) ds,$$

is bounded by

$$|I(\rho)| \leq C \begin{cases} \sinh^{-\frac{N-1}{2}} \rho, & \rho \geq 1, \\ 1, & 0 < \rho \leq 1. \end{cases}$$

If instead

$$(5.10) \quad |h(s)| \leq \begin{cases} s \sinh^{-\frac{N}{2}} s, & s \geq 1, \\ 1, & 0 < s \leq 1. \end{cases}$$

there holds that

$$|I(\rho)| \leq C \begin{cases} \rho \sinh^{-\frac{N-1}{2}} \rho, & \rho \geq 1, \\ 1, & 0 < \rho \leq 1. \end{cases}$$

*Proof.* We split the region of integration so that

$$I = \int_{\rho}^{\rho+2} \frac{\sinh s}{\sqrt{\cosh s - \cosh \rho}} h(s) ds + \int_{\rho+2}^{\infty} \frac{\sinh s}{\sqrt{\cosh s - \cosh \rho}} h(s) ds := I_1 + I_2.$$

First assume that  $0 \leq \rho \leq 1$ . Note that, by Taylor's expansion at the origin,

$$\cosh s - \cosh \rho = \frac{s^2 - \rho^2}{2!} + \frac{s^4 - \rho^4}{4!} + \dots,$$

so we can bound  $I_1$  by

$$\begin{aligned} |I_1| &\leq C \int_{\rho}^{\rho+2} \frac{\sinh s}{\sqrt{\cosh s - \cosh \rho}} ds \leq C \int_{\rho}^{\rho+2} \frac{s}{\sqrt{s^2 - \rho^2}} ds, \\ &= C \sqrt{(\rho+2)^2 - \rho^2} \leq 3C \end{aligned}$$

Next, we estimate  $I_2$ . First assume  $h$  satisfies (5.9) then, since  $0 < \rho \leq 1$ ,

$$|I_2(\rho)| \leq \int_{\rho+2}^{\infty} \frac{(\sinh s)^{1-\frac{N}{2}}}{\sqrt{\cosh s - \cosh \rho}} ds \leq C \int_2^{\infty} (\sinh s)^{-\frac{N-1}{2}} ds = C.$$

Now, assume that  $\rho \geq 1$ . First observe that

$$\cosh s - \cosh \rho \geq (s - \rho) \sinh \rho.$$

Therefore,

$$|I_1(\rho)| \leq \frac{1}{(\sinh \rho)^{1/2}} \int_{\rho}^{\rho+2} \frac{(\sinh s)^{1-\frac{N}{2}}}{\sqrt{s - \rho}} ds \leq \frac{1}{(\sinh \rho)^{\frac{N-1}{2}}} \int_0^2 \tau^{-1/2} d\tau = C \frac{1}{(\sinh \rho)^{\frac{N-1}{2}}}.$$

In order to bound  $I_2(\rho)$ , we observe from Taylor's expansion at  $s = \rho$  that for  $n$  odd and  $s > \rho$ ,

$$\cosh s - \cosh \rho \geq (\sinh \rho) \frac{(s - \rho)^n}{n!}$$

so that,

$$|I_2(\rho)| \leq \frac{1}{(\sinh \rho)^{1/2}} \int_{\rho+2}^{\infty} \frac{(\sinh s)^{1-\frac{N}{2}}}{(s-\rho)^{\frac{n}{2}}} ds.$$

By taking  $n > 2N$  we get,

$$|I_2(\rho)| \leq \frac{C}{(\sinh \rho)^{\frac{N-1}{2}}}.$$

Now, if  $h$  satisfies (5.10), we proceed similarly. By using that  $s \leq \rho + 2 \leq 3$  when  $0 < \rho \leq 1$ , the bound of  $I_1$  follows in that case. In order to bound  $I_2(\rho)$  for  $0 < \rho \leq 1$  we proceed as before and we get,

$$|I_2(\rho)| \leq C \int_2^{\infty} s(\sinh s)^{-\frac{N-1}{2}} ds = C.$$

Finally, for  $\rho > 1$ , since  $\frac{s}{s-\rho} \leq \frac{\rho+2}{2}$  for  $s \geq \rho + 2$ ,

$$\begin{aligned} |I_2(\rho)| &\leq \frac{1}{(\sinh \rho)^{1/2}} \int_{\rho+2}^{\infty} \frac{s(\sinh s)^{1-\frac{N}{2}}}{(s-\rho)^{\frac{n}{2}}} ds \\ &\leq C \frac{\rho}{(\sinh \rho)^{1/2}} \int_{\rho+2}^{\infty} \frac{(\sinh s)^{1-\frac{N}{2}}}{(s-\rho)^{\frac{n}{2}-1}} ds. \end{aligned}$$

And the result follows by taking  $n > 2(N+1)$ .  $\square$

**5.2. The heat equation on hyperbolic space.** The explicit expression for the heat kernel on hyperbolic space is well known in the literature (see, for instance, [20] and the references therein, or [14, 2]). However, we provide a direct proof using the Fourier transform in hyperbolic space, since some of the ingredients will be used in the proof of the main theorem.

**Proposition 5.1.** *The solution of*

$$(5.11) \quad \begin{cases} v_t = b\Delta_{\mathbb{H}^N} v + \left(a - 1 + b\frac{(N-1)^2}{4}\right)v & \text{in } \mathbb{H}^N \times (0, \infty), \\ v(x, 0) = u_0(x) & \text{in } \mathbb{H}^N, \end{cases}$$

may be written explicitly as

$$v(x, t) = e^{-(1-a)t} \int_{\mathbb{H}^N} u_0(x') K_0(s_{xx'}, t) d\mu_{x'},$$

where, for  $N \geq 3$  odd,

$$(5.12) \quad K_0(\rho, t) = C_N \frac{1}{\sqrt{bt}} \left( \frac{\partial_\rho}{\sinh \rho} \right)^{\frac{N-1}{2}} e^{-\frac{\rho^2}{4bt}},$$

and, for  $N \geq 2$  even,

$$K_0(\rho, t) = C_N \frac{1}{(bt)^{3/2}} \left( \frac{\partial_\rho}{\sinh \rho} \right)^{\frac{N-2}{2}} \int_\rho^\infty \frac{s e^{-\frac{s^2}{4bt}}}{\sqrt{\cosh s - \cosh \rho}} ds.$$

Moreover, these kernels satisfy the following estimates when  $t \rightarrow +\infty$ :

$$(5.13) \quad |K_0(\rho, t)| \leq C \frac{1}{t^{3/2}} \frac{\rho}{(\sinh \rho)^{\frac{N-1}{2}}}, \quad \rho \geq 1,$$



and,

$$(5.14) \quad |K_0(\rho, t)| \leq C \frac{1}{t^{3/2}}, \quad 0 < \rho \leq 1,$$

*Proof.* Taking Fourier transform, the solution of (5.11) may be written as

$$\hat{v}(\lambda, \omega, t) = \hat{u}_0(\lambda, \omega) e^{(a-1-b\lambda^2)t}.$$

Therefore, by formula (5.4) we can express this solution as

$$v(x, t) = e^{(a-1)t} \int_{\mathbb{H}^N} u_0(x') K_0(s_{xx'}, t) d\mu_{x'},$$

where

$$K_0(\rho, t) = \int_{-\infty}^{\infty} e^{-b\lambda^2 t} k_\lambda(\rho) d\lambda.$$

More precisely, for  $N$  odd, using (5.6) we can write

$$(5.15) \quad K_0(\rho, t) = c \left( \frac{\partial_\rho}{\sinh \rho} \right)^{\frac{N-1}{2}} \int_{-\infty}^{\infty} e^{-b\lambda^2 t} (\cos \lambda \rho) d\lambda.$$

But then we calculate

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-b\lambda^2 t} (\cos \lambda \rho) d\lambda &= \int_{-\infty}^{\infty} e^{-b\lambda^2 t + i\lambda \rho} d\lambda = \int_{-\infty}^{\infty} e^{-\left(\lambda\sqrt{bt} - \frac{i\rho}{2\sqrt{bt}}\right)^2} d\lambda \\ &= \frac{1}{\sqrt{bt}} e^{-\frac{\rho^2}{4bt}} \int_{-\infty}^{\infty} e^{-\left(\eta - \frac{i\rho}{2\sqrt{bt}}\right)^2} d\eta = \frac{2\sqrt{\pi}}{\sqrt{bt}} e^{-\frac{\rho^2}{4bt}}, \end{aligned}$$

which shows (5.12).

Now we prove the estimates (5.13) and (5.14) in the case that  $N$  is odd. First, we may use Lemma 5.2 to bound (5.15) near  $\rho = 0$ . Indeed, by Lemma 5.2, *i.* if  $\lambda\rho \leq 1$  and *ii.* if  $\lambda\rho > 1$  we have, for  $0 < \rho \leq 1$ ,

$$\left| \left( \frac{\partial_\rho}{\sinh \rho} \right)^{\frac{N-1}{2}} \cos \lambda \rho \right| \leq C \sum_{n=2}^{N-1} \lambda^n$$

so that, if  $t \geq 1$ ,

$$|K_0(\rho, t)| \leq C \sum_{n=2}^{N-1} \int_{-\infty}^{\infty} \lambda^n e^{-b\lambda^2 t} d\lambda = C \sum_{n=2}^{N-1} \frac{1}{(bt)^{\frac{n+1}{2}}} \int_{-\infty}^{\infty} \eta^n e^{-\eta^2} d\eta \leq \frac{C}{t^{3/2}}.$$

Now, for  $\rho > 1$ , by Corollary 1.1,

$$\left| \left( \frac{\partial_\rho}{\sinh \rho} \right)^{\frac{N-1}{2}} \cos \lambda \rho \right| \leq C \frac{\rho}{(\sinh \rho)^{\frac{N-1}{2}}} \sum_{n=2}^{\frac{N-1}{2}} \lambda^n$$

and the same computation as above gives (5.13).

On the other hand, for  $N$  even

$$K_0(\rho, t) = c \int_{-\infty}^{\infty} e^{-b\lambda^2 t} \int_{\rho}^{\infty} \frac{\sinh s}{\sqrt{\cosh s - \cosh \rho}} \left( \frac{\partial_s}{\sinh s} \right)^N \cos \lambda s ds d\lambda.$$

By Lemma 1.2 and Corollary 1.1, we have,

$$K_0(\rho, t) = c \sum_{n=2}^N I_n(\rho) \int_{-\infty}^{\infty} \lambda^n e^{-b\lambda^2 t} d\lambda$$

where

$$I_n(\rho) = \int_{\rho}^{\infty} \frac{\sinh s}{\sqrt{\cosh s - \cosh \rho}} h_n(s) ds$$

with

$$|h_n(s)| \leq C \begin{cases} 1 & 0 < s \leq 1, \\ \frac{s}{(\sinh s)^{\frac{N}{2}}} & s > 1. \end{cases}$$

Hence, by Lemma 1.4 and the computations above, estimates (5.13) and (5.14) hold.  $\square$

**Remark 5.2.** Observe from the proof of Proposition 2.1 that for  $0 < t \leq 1$  the power  $t^{-3/2}$  in the estimates of  $K_0$  is replaced by  $t^{-N/2}$ , but the  $t^{-3/2}$  decay cannot be improved for  $t$  large. In the Discussion section at the end we will provide some heuristic arguments explaining this exponent.

**5.3. Asymptotic profile for the non-local equation.** We are given  $J \in C_0^\infty(\mathbb{H}^N)$  a radially symmetric kernel. Consider the Cauchy problem

$$(5.16) \quad \begin{cases} u_t = J * u - u & \text{in } \mathbb{H}^N \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{on } \mathbb{H}^N. \end{cases}$$

Recall that the convolution is defined as

$$J * u(x) = \int_{\mathbb{H}^N} u(x') J(s_{xx'}) d\mu_{x'}.$$

Note that  $\hat{J} = \hat{J}(\lambda)$  is a radially symmetric function that has an entire extension to  $\mathbb{C}$  and decays at infinity faster than any negative power of  $|\lambda|$ . Moreover, the asymptotic behavior of the solution to (5.16) as  $t \rightarrow \infty$  is governed by the behavior of  $\hat{J}(\lambda)$  at the origin. In fact, we have the expansion

$$\hat{J}(\lambda) = a - b\lambda^2 + \lambda^2 f(\lambda),$$

for some  $f(\lambda) = O(\lambda^2)$  an even function (note that  $J$  being a radially symmetric kernel has even Fourier transform). These coefficients are given by the formulas

$$\begin{aligned} a &:= \int_{\mathbb{H}^N} J(s_{xO}) \Phi_0(x) d\mu_x, \\ b &:= -\frac{1}{2} \int_{\mathbb{H}^N} J(s_{xO}) \partial_{\lambda\lambda}|_{\lambda=0} \Phi_\lambda(x) d\mu_x. \end{aligned}$$

Observe that  $b > 0$ . In fact,

$$\partial_{\lambda\lambda} \Phi_{-\lambda}(\rho) \Big|_{\lambda=0} = -c_N \int_0^\pi (\cosh \rho - \sinh \rho \cos \theta)^{-\frac{N-1}{2}} \log^2(\cosh \rho - \sinh \rho \cos \theta) d\theta < 0$$

for every  $\rho > 0$ . Hence,

$$b = -\partial_{\lambda\lambda} \hat{J}(0) = \int_0^\infty J(\rho) \partial_{\lambda\lambda} \Phi_{-\lambda}(\rho) \Big|_{\lambda=0} (\sinh \rho)^{N-1} d\rho < 0.$$

We will compare to a translated heat equation on hyperbolic space with the same initial condition

$$(5.17) \quad \begin{cases} v_t = b \Delta_{\mathbb{H}^N} v + \left( a - 1 + b \frac{(N-1)^2}{4} \right) v = 0 & \text{in } \mathbb{H}^N \times (0, \infty), \\ v(x, 0) = u_0(x) & \text{in } \mathbb{H}^N, \end{cases}$$

according to Proposition 5.1, we can write explicitly the solution to this equation. Indeed,

$$v(x, t) = e^{-(1-a)t} \int_{\mathbb{H}^N} u_0(x') K_0(s_{xx'}, t) d\mu_{x'},$$

where  $K_0$  is given explicitly in the proposition and it satisfies the estimates for  $t$  large

$$\begin{aligned} |K_0(\rho, t)| &\leq C \frac{1}{t^{3/2}} \frac{\rho}{(\sinh \rho)^{\frac{N-1}{2}}}, \quad \rho \geq 1, \\ |K_0(\rho, t)| &\leq C \frac{1}{t^{3/2}}, \quad 0 < \rho \leq 1. \end{aligned}$$

Before we state the main result of this section let us prove the following fact:

**Lemma 5.5.** *Let  $u_0$  be such that the function*

$$v(x) := \sup_{s_{yO}=s_{xO}} |u_0(y)|$$

*belongs to  $L^1(\mathbb{H}^N)$ . Then,  $\hat{u}_0 \in L^\infty(\mathbb{R} \times \mathbb{S}^{N-1})$ .*

*Proof.* There holds,

$$\begin{aligned} |\hat{u}_0(\lambda, \omega)| &\leq \int_0^\infty \int_{\mathbb{S}^{N-1}} |u_0(\cosh \rho, \sigma \sinh \rho)| |h_{-\lambda, \omega}(\cosh \rho, \sigma \sinh \rho)| (\sinh \rho)^{N-1} d\sigma d\rho \\ &\leq \int_0^\infty v(\rho) \int_{\mathbb{S}^{N-1}} |h_{\lambda, \omega}(\cosh \rho, \sigma \sinh \rho)| d\sigma (\sinh \rho)^{N-1} d\rho. \end{aligned}$$

The observation that, by [26], Lemma 2.2 (a),

$$\int_{\mathbb{S}^{N-1}} |h_{-\lambda, \omega}(\cosh \rho, \sigma \sinh \rho)| d\omega = \Phi_0(x) \leq C,$$

ends the proof.  $\square$

Now we state our main result, that is the profile when  $t \rightarrow +\infty$  of the convolution equation is the same one as the one for the translated Heat-Beltrami equation:

**Theorem 5.1.** *Let  $u$  be the solution of (5.16) and  $v$  the solution of (5.17), with initial condition  $u_0 \in L^\infty(\mathbb{H}^N) \cap L^1(\mathbb{H}^N)$  such that  $\hat{u}_0 \in L^\infty(\mathbb{R} \times \mathbb{S}^{N-1})$  (observe that this is the case if, for instance,  $u_0$  is radially symmetric).*

*Then*

$$\lim_{t \rightarrow \infty} e^{(1-a)t} t^{3/2} \sup_{x \in \mathbb{H}^N} |u(x, t) - v(x, t)| = 0.$$

*Proof.* Taking Fourier transform and using the properties of the convolution on hyperbolic space, the solutions of (5.16) and (5.17) may be written, respectively, as

$$\hat{u}(\lambda, \omega, t) = \hat{u}_0(\lambda, \omega) e^{(\hat{J}(\lambda)-1)t}$$

and

$$\hat{v}(\lambda, \omega, t) = \hat{u}_0(\lambda, \omega) e^{(a-1-b\lambda^2)t}.$$

Looking at the expressions for  $\hat{u}$  and  $\hat{v}$ , from (5.4) we can conclude that

$$\begin{aligned} u(x, t) - v(x, t) &= \int_{\mathbb{H}^N} \int_{-\infty}^\infty \left[ e^{(\hat{J}(\lambda)-1)t} - e^{(a-1-b\lambda^2)t} \right] k_\lambda(s_{xx'}) u_0(x') d\lambda d\mu_{x'} \\ &= e^{(a-1)t} \int_{\mathbb{H}^N} K(s_{xx'}, t) u_0(x') d\mu_{x'}, \end{aligned}$$

where

$$K(\rho, t) = \int_{-\infty}^{\infty} \left[ e^{(-b\lambda^2 + \lambda^2 f(\lambda))t} - e^{-b\lambda^2 t} \right] k_\lambda(\rho) d\lambda.$$

In order to get the result, we split the integral into two regions:

$$K_1(\rho, t) := \int_{|\lambda| \geq \tau(t)} \left[ e^{(-b\lambda^2 + \lambda^2 f(\lambda))t} - e^{-b\lambda^2 t} \right] k_\lambda(\rho) d\lambda$$

and

$$K_2(\rho, t) := \int_{|\lambda| < \tau(t)} \left[ e^{(-b\lambda^2 + \lambda^2 f(\lambda))t} - e^{-b\lambda^2 t} \right] k_\lambda(\rho) d\lambda$$

for some  $\tau(t)$  chosen as  $\tau(t) = t^{-1/2+\varepsilon}$  for some  $\varepsilon$  small enough. We claim that, both in the even and in the odd case,

$$(5.18) \quad \begin{aligned} t^{3/2} \sup_{0 < \rho \leq 1} \{|K_2(\rho, t)|\} &\rightarrow 0, \quad \text{as } t \rightarrow +\infty, \\ t^{3/2} \sup_{\rho \geq 1} \left\{ (\sinh \rho)^{\frac{N-1}{2}} |K_2(\rho, t)| \right\} &\rightarrow 0, \quad \text{as } t \rightarrow +\infty. \end{aligned}$$

Now we prove this claim. Let us start with  $N$  odd. In this case

$$k_\lambda(\rho) = c \left( \frac{\partial_\rho}{\sinh \rho} \right)^{\frac{N-1}{2}} \cos \lambda \rho$$

so that, by Lemma 1.2 and Remark 1.1, if  $\varepsilon$  is small enough,

$$\begin{aligned} |K_2(\rho, t)| &\leq \int_{|\lambda| < \tau(t)} e^{-b\lambda^2 t} |e^{\lambda^2 f(\lambda)t} - 1| \left| \left( \frac{\partial_\rho}{\sinh \rho} \right)^{\frac{N-1}{2}} \cos \lambda \rho \right| d\lambda \\ &\leq C \int_{|\lambda| < \tau(t)} e^{-b\lambda^2 t} \lambda^2 f(\lambda) t \sum_{n=1}^{N-1} \lambda^n d\lambda \begin{cases} 1 & 0 < \rho \leq 1, \\ \frac{1}{(\sinh \rho)^{\frac{N-1}{2}}} & \rho > 1 \end{cases} \end{aligned}$$

Notice that,

$$\begin{aligned} t \int_{|\lambda| < \tau(t)} e^{-b\lambda^2 t} \lambda^{n+2} f(\lambda) d\lambda &\leq Ct \int_{|\lambda| < \tau(t)} e^{-b\lambda^2 t} \lambda^{n+4} d\lambda \\ &= \frac{Ct}{(bt)^{\frac{n+5}{2}}} \int_{|\eta| < \sqrt{bt}\tau(t)} \eta^{n+4} e^{-\eta^2} d\eta \leq \frac{C}{t^{\frac{n+3}{2}}}. \end{aligned}$$

Therefore, (5.18) holds.

If  $N$  is even we obtain similar bounds by using Lemma 5.4. In fact,

$$|K_2(\rho, t)| \leq I_n(\rho) \sum_{n=1}^N \int_{|\lambda| < \tau(t)} e^{-b\lambda^2 t} |e^{\lambda^2 f(\lambda)t} - 1| \lambda^n d\lambda$$

where

$$I_n(\rho) = \int_\rho^\infty \frac{\sinh s}{\sqrt{\cosh s - \cosh \rho}} h_n(s) ds$$

with

$$0 \leq h_n(s) \leq c \begin{cases} 1 & 0 < s \leq 1, \\ \frac{1}{(\sinh s)^{\frac{N}{2}}} & s \geq 1. \end{cases}$$

Now, in order to estimate

$$\int_{\mathbb{H}^N} K_1(s_{xx'}, t) u_0(x') d\mu_{x'}$$

we split the kernel  $K_1$  as  $K_1 = K_{11} + K_{12}$  with

$$(5.19) \quad K_{11}(\rho, t) = - \int_{|\lambda| \geq \tau(t)} e^{-b\lambda^2 t} k_\lambda(\rho) d\lambda.$$

and

$$K_{12}(\rho, t) = \int_{|\lambda| \geq \tau(t)} e^{(\hat{J}(\lambda)-a)t} k_\lambda(\rho) d\lambda.$$

First we turn our attention to (5.19). In the case  $N$  is odd, proceeding as before, we get for the interval  $0 < \rho \leq 1$ , that

$$|K_{11}(\rho, t)| \leq C \sum_{n=2}^{N-1} \int_{|\lambda| > \tau(t)} e^{-b\lambda^2 t} \lambda^n d\lambda = C \sum_{n=2}^{N-1} \frac{1}{(bt)^{\frac{n+1}{2}}} \int_{|\eta| > b^{1/2} t^\varepsilon} \eta^n e^{-\eta^2} d\eta.$$

So that,

$$t^{3/2} \sup_{0 < \rho \leq 1} |K_{11}(\rho, t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Now, if  $\rho > 1$ , by Remark 1.1,

$$\begin{aligned} |K_{11}(\rho, t)| &\leq \frac{C}{(\sinh \rho)^{\frac{N-1}{2}}} \sum_{n=1}^{\frac{N-1}{2}} \int_{|\lambda| > \tau(t)} e^{-b\lambda^2 t} \lambda^n d\lambda \\ &= \frac{C}{(\sinh \rho)^{\frac{N-1}{2}}} \sum_{n=1}^{\frac{N-1}{2}} \frac{1}{(bt)^{\frac{n+1}{2}}} \int_{|\eta| > b^{1/2} t^\varepsilon} \eta^n e^{-\eta^2} d\eta \\ &\leq \frac{C}{(\sinh \rho)^{\frac{N-1}{2}}} e^{-\frac{1}{2} b^{1/2} t^\varepsilon} \sum_{n=1}^{\frac{N-1}{2}} \frac{1}{(bt)^{\frac{n+1}{2}}} \int \eta^n e^{-\eta^2/2} d\eta. \end{aligned}$$

And we get

$$t^{3/2} \sup_{\rho \geq 1} (\sinh \rho)^{\frac{N-1}{2}} |K_{11}(\rho, t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Finally, we turn to the kernel  $K_{12}$ . We split it again as

$$K_{12} = K_{121} + K_{122}$$

where

$$K_{121}(\rho, t) = \int_{\tau(t) < |\lambda| \leq R} e^{(\hat{J}(\lambda)-a)t} k_\lambda(\rho) d\lambda$$

and

$$K_{122}(\rho, t) = \int_{|\lambda| > R} e^{(\hat{J}(\lambda)-a)t} k_\lambda(\rho) d\lambda.$$

We choose  $R$  small so that,  $\hat{J}(\lambda) - a \leq -\tilde{b}\lambda^2$  if  $|\lambda| \leq R$  with  $\tilde{b} > 0$ . Then, since  $\hat{J}$  is analytic,  $\hat{J}(\lambda) \rightarrow 0$  as  $|\lambda| \rightarrow \infty$  and  $\hat{J}(\lambda) < a - \delta$  if  $|\lambda| = R$  for some positive  $\delta$ , there holds that  $\hat{J}(\lambda) - a \leq -\delta$  if  $|\lambda| > R$ .

Let us consider the term  $K_{121}$ . There holds,

$$K_{121}(\rho, t) \leq \int_{\tau(t) < |\lambda| \leq R} e^{-\tilde{b}\lambda^2 t} k_\lambda(\rho) d\lambda.$$

Proceeding as above we see that (5.18) holds with  $K_2$  replaced by  $K_{121}$ .

We have to treat the term  $K_{122}$  in a different way. So, before we proceed let us see what we have up to now.

$$|u(x, t) - v(x, t)| \leq e^{-(1-a)t} \int_{\mathbb{H}^N} |\tilde{K}(s_{xx'}, t)| |u_0(x')| d\mu_{x'} + e^{-(1-a)t} \left| \int_{\mathbb{H}^N} K_{122}(s_{xx'}, t) u_0(x') d\mu_{x'} \right|$$

where  $\tilde{K} = K - K_{122}$ . By the estimates we have already proven,

$$\begin{aligned} t^{3/2} \int_{\mathbb{H}^N} |\tilde{K}(s_{xx'}, t)| |u_0(x')| d\mu_{x'} &\leq o_t(1) \int_{\{s_{xx'} \leq 1\}} |u_0(x')| d\mu_{x'} \\ &+ o_t(1) \int_{\{s_{xx'} \geq 1\}} \frac{|u_0(x')|}{(\sinh s_{xx'})^{\frac{N-1}{2}}} d\mu_{x'} \leq o_t(1), \end{aligned}$$

under our assumptions for the initial data  $u_0$ .

In order to finish the proof, let us estimate

$$A = \int_{\mathbb{H}^N} K_{122}(s_{xx'}, t) u_0(x') d\mu_{x'}.$$

There holds,

$$\begin{aligned} A &= \int_{\mathbb{H}^N} \int_{|\lambda| > R} e^{(j(\lambda)-a)t} \int_{\mathbb{S}^{N-1}} h_{\lambda, \omega}(x) h_{-\lambda, \omega}(x') \frac{d\omega}{|c(\lambda)|^2} d\lambda u_0(x') d\mu_{x'} \\ &= \int_{\mathbb{S}^{N-1}} \int_{|\lambda| > R} e^{(j(\lambda)-a)t} \frac{h_{\lambda, \omega}(x)}{|c(\lambda)|^2} \int_{\mathbb{H}^N} h_{-\lambda, \omega}(x') u_0(x') d\mu_{x'} d\lambda d\omega \\ &= \int_{\mathbb{S}^{N-1}} \int_{|\lambda| > R} e^{(j(\lambda)-a)t} h_{\lambda, \omega}(x) \hat{u}_0(\lambda, \omega) \frac{d\omega d\lambda}{|c(\lambda)|^2} \\ &= e^{-at} \int_{\mathbb{S}^{N-1}} \int_{|\lambda| > R} h_{\lambda, \omega}(x) \hat{u}_0(\lambda, \omega) \frac{d\omega d\lambda}{|c(\lambda)|^2} \\ &\quad + \int_{\mathbb{S}^{N-1}} \int_{|\lambda| > R} e^{-at} (e^{j(\lambda)t} - 1) h_{\lambda, \omega}(x) \hat{u}_0(\lambda, \omega) \frac{d\omega d\lambda}{|c(\lambda)|^2} = A_1 + A_2 \end{aligned}$$

Using that by hypothesis both  $u_0$  and  $\hat{u}_0$  are bounded, the fact that

$$(5.20) \quad \int_{\mathbb{S}^{N-1}} |h_{\lambda, \omega}(x)| d\omega = \Phi_0(x) \leq C$$

by (5.2), the estimate

$$|c(\lambda)|^{-2} \leq C\lambda^2 \quad \text{for } \lambda \text{ bounded,}$$

and the formula

$$A_1 = e^{-at} u_0(x) - e^{-at} \int_{\mathbb{S}^{N-1}} \int_{|\lambda| \leq R} h_{\lambda, \omega}(x) \hat{u}_0(\lambda, \omega) \frac{d\omega d\lambda}{|c(\lambda)|^2},$$

it follows that,

$$t^{3/2} |A_1(x, t)| \leq o_t(1) \quad \text{when } t \rightarrow +\infty.$$

On the other hand, using  $|c(\lambda)|^{-2} \leq C_R |\lambda|^{N-1}$  for  $|\lambda| > R$ , the same formula (5.20), and the fact that  $\hat{J}$  decays faster than any negative power of  $|\lambda|$  we get,

$$\begin{aligned} |A_2(x, t)| &\leq t e^{-\delta t} \int_{\mathbb{S}^{N-1}} \int_{|\lambda| > R} |\hat{J}(\lambda)| |h_{\lambda, \omega}(x)| |\hat{u}_0(\lambda, \omega)| \frac{d\omega d\lambda}{|c(\lambda)|^2} \\ &\leq C t e^{-\delta t} \int_{|\lambda| > R} |\lambda|^{N-1} |\hat{J}(\lambda)| d\lambda \leq C t e^{-\delta t}. \end{aligned}$$

and the theorem is proved.  $\square$

**5.4. Discussion.** From the previous subsections we find that, in order for the nonlocal problem to behave as the classical translated diffusion equation on hyperbolic space, namely as

$$(5.21) \quad v_t = b \left( \Delta_{\mathbb{H}^N} + \frac{(N-1)^2}{4} \right) v$$

the kernel  $J$  should be normalized so that

$$(5.22) \quad a = \int_{\mathbb{H}^N} J(s_{xO}) \Phi_0(x) d\mu_x = 1.$$

This normalization differs from the one that preserves mass. This is, to normalize  $J$  so that  $\int_{\mathbb{H}^N} J(s_{xO}) d\mu_x = 1$  (as a consequence of Fubini's Theorem this normalization implies mass preservation).

In Euclidean space, mass preservation and preservation of  $\hat{u}(0, t)$  are two ways of stating the same fact. Let us show that the normalization (5.22) implies that

$$\int_{\mathbb{H}^N} u(x, t) h_{0, \omega}(x) d\mu_x = \hat{u}(0, \omega, t)$$

is preserved for every  $\omega \in \mathbb{S}^{N-1}$ .

In fact,

$$\int_{\mathbb{H}^N} (J * u(\cdot, t))(x) h_{0, \omega}(x) d\mu_x = \left( \int_{\mathbb{H}^N} J(s_{xO}) h_{0, \omega}(x) d\mu_x \right) \left( \int_{\mathbb{H}^N} u(x, t) h_{0, \omega}(x) d\mu_x \right).$$

The proof of this identity follows simply by considering the property (5.3) of the convolution on hyperbolic space –using that  $J$  is radially symmetric– and evaluating the Fourier transforms that appear at  $\lambda = 0$ .

As a consequence, if we impose the normalization (5.22), by multiplying equation (5.16) by  $h_{0, \omega}(x)$  and integrating on  $\mathbb{H}^N$  we get

$$\frac{d}{dt} \int_{\mathbb{H}^N} u(x, t) h_{0, \omega}(x) d\mu_x = \int_{\mathbb{H}^N} (J * u(\cdot, t))(x) h_{0, \omega}(x) d\mu_x - \int_{\mathbb{H}^N} u(x, t) h_{0, \omega}(x) d\mu_x = 0.$$

This is, there is conservation of the integral

$$(5.23) \quad \int_{\mathbb{H}^N} u(x, t) h_{0, \omega}(x) d\mu_x$$

for every  $\omega \in \mathbb{S}^{N-1}$ . In addition, integrating on  $\omega \in \mathbb{S}^{N-1}$  we have that

$$(5.24) \quad \int_{\mathbb{H}^N} u(x, t) \Phi_0(x) d\mu_x$$

is also preserved.

Let us give a closer look at the Heat-Beltrami equation on hyperbolic space. First, observe that (5.23) and (5.24) are the natural quantities that are preserved for the classical diffusion equation (5.21) on hyperbolic space (for the proof, just take Fourier transform and evaluate at  $\lambda = 0$ ).

On the other hand, let us define the quantity

$$P(r, \theta, t) = v(r, \theta, t) \Phi_0(r) (\sinh r)^{N-1}.$$

Observe also that

$$\int_{\mathbb{H}^N} v(x, t) \Phi_0(x) d\mu_x = \int_0^\infty \int_{\mathbb{S}^{N-1}} P(r, \theta, t) dr d\theta.$$

A straightforward calculation yields, from (5.21), the equation

$$(5.25) \quad P_t = \partial_r \left( P_r - \left[ (N-1) \frac{\cosh r}{\sinh r} + 2\partial_r \log \Phi_0 \right] P \right) + \frac{1}{\sinh^2 r} \Delta_{\mathbb{S}^{N-1}} P,$$

from where we deduce again that  $\int_{\mathbb{H}^N} v(x, t) \Phi_0(x) d\mu_x$  is constant in time.

In equation (5.25) we can recognize a drift term with velocity

$$V(r) = (N-1) \frac{\cosh r}{\sinh r} + 2\partial_r \log \Phi_0,$$

and, by the asymptotics  $\Phi_0(r) = e^{-\frac{N-1}{2}r} (r + o(r))$  as  $r \rightarrow \infty$ , we find that  $V(r) = \frac{2}{r} + o(r^{-1})$ . Since the angular operator in (5.25) is multiplied by an exponentially decaying factor, we can expect that the right hand side of (5.25) is, at leading order,

$$\partial_r \left( P_r + \frac{2}{r} P \right) = \frac{1}{r^2} \partial_r (r^2 P_r),$$

which is the Euclidean Laplacian for radial functions in dimension 3. It is well known that the decay of the fundamental solution for the corresponding heat equation is  $t^{-3/2}$ , which motivates the results of the Subsection 5.2 about the decay of the diffusion on hyperbolic space.

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