SCALING LIMITS FOR THE CRITICAL FORTUIN-KASTELYN MODEL ON A RANDOM PLANAR MAP II: LOCAL ESTIMATES AND EMPTY REDUCED WORD EXPONENT

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Abstract. We continue our study of the inventory accumulation introduced by Sheffield (2011), which encodes a random planar map decorated by a collection of loops sampled from the critical Fortuin-Kastelyn (FK) model. We prove various "local" estimates for the inventory accumulation model, i.e. estimates for the precise number of symbols of a given type in a word sampled from the model. Using our estimates, we obtain the scaling limit of the associated two-dimensional random walk conditioned on the event that it stays in the first quadrant for one unit of time and ends up at a particular position in the interior of the first quadrant. We also obtain the exponent for the probability that a word of length $2n$ sampled from the inventory accumulation model corresponds to an empty reduced word. The estimates of this paper will be used in a subsequent paper to obtain the scaling limit of the random walk associated with a finite-volume FK planar map.

Contents

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1. Introduction

In [She11], Sheffield introduces a method to encode a *(critical) FK planar map*, i.e. a pair consisting of a random planar map M and a set S of edges on M sampled from the uniform distribution on such pairs weighted by the partition function of the critical Fortuitn-Kastelyn (FK) model, by certain words in an alphabet of five symbols. This word, in turn, gives rise to a two-dimensional lattice walk. This encoding is called the *hamburger-cheeseburger bijection* because it has a natural interpretation as an inventory accumulation process of a certain burger restaurant. The hamburger-cheeseburger bijection generalizes a bijection due to Mullin [Mul67] (see also [Ber07]) and is equivalent for a fixed choice of M to the bijection described in [Ber08, Section 4].

In addition to its interest as a tool for studying FK planar maps, the hamburger-cheeseburger bijection serves as the main source of discrete intuition behind the recent work [DMS14] which introduce the "peanosphere construction" to encode a conformal loop ensemble (CLE_κ) on a Liouville quantum gravity (LQG) surface with parameter $\gamma = 4/\sqrt{\kappa}$. The main result in [She11] is that the lattice walk constructed from the word associated with an infinite-volume FK-weighted random planar map converges in the scaling limit to a positively correlated two-sided Brownian motion, which is the same Brownian motion appeared in the Peano sphere theory of the corresponding LQG in [DMS14]. In this sense, [She11] can be seen as a scaling limit result from FK planar maps to the corresponding LQG in a certain topology.

There have been several recent works regarding the hamburger-cheeseburger approach to critical FK planar maps. In the article [GMS15] the present authors and C. Mao improved the topology in the convergence to LQG in [She11] by proving a statement which implies, among other things, the convergence of the quantum areas and quantum lengths associated with macroscopic FK loops. The authors of [BLR15] identified the tail exponents for the laws of several quantities associated with FK loops ([GMS15] independently proves that the tail is actually regularly varying with the same for several of these quantities). The work [Che15] studies the whole plane version of the hamburger-cheeseburger bijection.

We note that in the special case of a uniform planar map (without loop decoration), there is also another approach based on the bijection of Schaeffer [Sch97], which has met with substantial success. We refer the reader to [Mie09, Le 14] and the references therein for more on this approach.

In this paper, we continue the theme of [GMS15] by studying the fine asymptotic properties of the word associated with critical FK planar maps. In particular, we will prove a variety of "local estimates" which give us up-to-constants asymptotics for the probability that a word sampled from the inventory accumulation model contains a *particular* number of symbols of each type. Such estimates play a crucial role in the study of small-scale events associated with the inventory accumulation model, e.g. the event that the associated lattice walk ends up at a particular point after a given amount of time. Local estimates are not proven in the works [She11, GMS15, BLR15], which focus mainly on the behavior of the word at large scales. The starting point of the proofs of our local estimates is the bivariate local limit theorem of Doney [Don91].

As an application of our estimates, in Theorem 1.8 we will prove that if we condition on the lattice walk in the hamburger-cheeseburger model to stay in the first quadrant for a certain amount of time and end up at a fix interior point, then the walk converges in the scaling limit to a correlated Brownian bridge conditioned on staying in the first quadrant. As another application, in Theorem 1.10 we obtain the exact exponent of the probability that a word of length $2n$ reduces to the empty word. The estimates established in this paper will also be used in the forthcoming work [GS15], in which we will prove analogues of the scaling limit results of [She11, GMS15] for the finite-volume version of the model in [She11] (which is encoded by a word of length $2n$ conditioned to reduce to the empty word); and in [GM15], in which the first author and J. Miller will prove convergence of the full topological structure of FK planar maps to that of a conformal loop ensemble on an independent Liouville quantum gravity surface.

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1.1. **Notation.** In this section we will introduce some notation which will remain fixed throughout the paper. This notation is in agreement with that used in [GMS15].

1.1.1. *Basic notation.*

Definition 1.1. Let X be a random variable taking values in a finite space Ω. A *realization* of X is an element $x \in \Omega$ such that $\mathbb{P}(X = x) > 0$.

Notation 1.2. *For $a < b \in \mathbb{R}$, we define the discrete intervals $[a,b]_{\mathbb{Z}} := [a,b] \cap \mathbb{Z}$ and $(a,b)_{\mathbb{Z}} := (a,b) \cap \mathbb{Z}$.*

Notation 1.3. *If a and b are two quantities, we write $a \preceq b$ (resp. $a \succeq b$) if there is a constant C (independent of the parameters of interest) such that $a \leq Cb$ (resp. $a \geq Cb$). We write $a \asymp b$ if $a \preceq b$ and $a \succeq b$.*

Notation 1.4. *If a and b are two quantities which depend on a parameter x, we write $a = o_x(b)$ (resp. $a = O_x(b)$) if $a/b \to 0$ (resp. a/b remains bounded) as $x \to 0$ (or as $x \to \infty$, depending on context). We write $a = o_x^\infty(b)$ if $a = o_x(b^s)$ for each $s \in \mathbb{R}$.*

Unless otherwise stated, all implicit constants in $\asymp, \preceq$, and $\succeq$ and $O_x(\cdot)$ and $o_x(\cdot)$ errors involved in the proof of a result are required to satisfy the same dependencies as described in the statement of said result.

1.1.2. *Inventory accumulation model.* Let $p \in (0, 1/2)$. We will always treat p as fixed and do not make dependence on p explicit. As explained in [She11, Section 4.2], the parameter p corresponds to an FK-weighted map of parameter $q = 4p^2/(1-p)^2$, which is conjectured to converge in the scaling limit to a CLE_κ with $\kappa \in (4, 8)$ satisfying

$$p = \frac{\sqrt{2 + 2\cos(8\pi/\kappa)}}{2 + \sqrt{2 + 2\cos(8\pi/\kappa)}}. \tag{1}$$

Let $\Theta := \{\text{H}, \text{C}, \boxed{\text{H}}, \boxed{\text{C}}, \boxed{\text{F}}\}$. We view elements of Θ as representing a hamburger, a cheeseburger, and hamburger order, a cheeseburger order, and a flexible order, respectively. The set Θ generates semigroup, which consists of the set of all finite words in elements of Θ, modulo the relations

$$\text{C}\boxed{\text{C}} = \text{H}\boxed{\text{H}} = \text{C}\boxed{\text{F}} = \text{H}\boxed{\text{F}} = \emptyset \quad \text{(order fulfilment)} \tag{2}$$

and

$$\text{C}\boxed{\text{H}} = \boxed{\text{H}}\text{C}, \qquad \text{H}\boxed{\text{C}} = \boxed{\text{C}}\text{H} \quad \text{(commutativity)}. \tag{3}$$

Given a word x consisting of elements of Θ, we denote by $\mathcal{R}(x)$ the word reduced modulo the above relations, with all burgers to the right of all orders. We also write $|x|$ for the number of symbols in x. We definite a probability measure on Θ by

$$\mathbb{P}\left(\text{H}\right) = \mathbb{P}\left(\text{C}\right) = \frac{1}{4}, \quad \mathbb{P}\left(\boxed{\text{H}}\right) = \mathbb{P}\left(\boxed{\text{C}}\right) = \frac{1-p}{4}, \quad \mathbb{P}\left(\boxed{\text{F}}\right) = \frac{p}{2}. \tag{4}$$

Let $X = \dots X_{-1}X_0X_1\dots$ be an infinite word with each symbol sampled independently according to the probabilities (4). For $a \leq b \in \mathbb{R}$, let

$$X(a,b) := \mathcal{R}\left(X_{\lfloor a \rfloor} \dots X_{\lfloor b \rfloor}\right). \tag{5}$$

We adopt the convention that $X(a,b) = \emptyset$ if $b < a$.

By [She11, Proposition 2.2], it is a.s. the case that the "infinite reduced word" $X(-\infty, \infty)$ is empty, i.e. each symbol X_i in the word X has a unique match which cancels it out in the reduced word.

Notation 1.5. *For $i \in \mathbb{Z}$ we write $\phi(i)$ for the index of the match of X_i.*

Notation 1.6. *For $\theta \in \Theta$ and a word x consisting of elements of Θ, we write $\mathcal{N}_\theta(x)$ for the number of θ-symbols in x. We also let*

$$d(x) := \mathcal{N}_{\text{(H)}}(x) - \mathcal{N}_{\boxed{H}}(x), \quad d^*(x) := \mathcal{N}_{\text{(C)}}(x) - \mathcal{N}_{\boxed{C}}(x), \quad D(x) := (d(x), d^*(x)).$$

The reason for the notation d and d^* is that these functions give the distances in the tree and dual tree which encode the collection of loops in the bijection of [She11, Section 4.1].

For $i \in \mathbb{Z}$, we define $Y_i = X_i$ if $X_i \in \{\text{H}, \text{C}, \boxed{\text{H}}, \boxed{\text{C}}\}$; $Y_i = \boxed{\text{H}}$ if $X_i = \boxed{\text{F}}$ and $X_{\phi(i)} = \text{H}$; and $Y_i = \boxed{\text{C}}$ if $X_i = \boxed{\text{F}}$ and $X_{\phi(i)} = \text{C}$. For $a \leq b \in \mathbb{R}$, define $Y(a,b)$ as in (5) with Y in place of X.

For $n \geq 0$, define $d(n) = d(Y(1,n))$ and for $n < 0$, define $d(n) = -d(Y(n+1,0))$. Define $d^*(n)$ similarly. Extend each of these functions from $\mathbb{Z}$ to $\mathbb{R}$ by linear interpolation. Let

$$D(t) := (d(t), d^*(t)). \tag{6}$$

For $n \in \mathbb{N}$ and $t \in \mathbb{R}$, let

$$U^n(t) := n^{-1/2} d(nt), \quad V^n(t) := n^{-1/2} d^*(nt), \quad Z^n(t) := (U^n(t), V^n(t)). \tag{7}$$

We note that the condition that $X(1,2n) = \emptyset$ is equivalent to the condition that $Z^n([0,2]) \subset [0,\infty)^2$ and $Z^n(2) = 0$.

Let $Z = (U,V)$ be a two-sided two-dimensional Brownian motion with $Z(0) = 0$ and variances and covariances at each time $t \in \mathbb{R}$ given by

$$\mathrm{Var}(U(t)) = \frac{1-p}{2}|t| \quad \mathrm{Var}(V(t)) = \frac{1-p}{2}|t| \quad \mathrm{Cov}(U(t), V(t)) = \frac{p}{2}|t|. \tag{8}$$

It is shown in [She11, Theorem 2.5] that as $n \to \infty$, the random paths Z^n defined in (7) converge in law in the topology of uniform convergence on compacts to the random path Z of (8).

There are several stopping times for the word X, read backward, which we will use throughout this paper. Namely, let

$$I := \inf\{i \in \mathbb{N} : X(1,i) \text{ contains an order}\}, \tag{9}$$

so that $\{I > n\}$ is the event that $X(-j,-1)$ contains a burger. For $m \in \mathbb{N}$, let

$$J_m^H := \inf\left\{ j \in \mathbb{N} : \mathcal{N}_{\text{(H)}}(X(-j,-1)) = m \right\}, \quad L_m^H := d^*\left(X(-J_m^H, -1)\right) \tag{10}$$

be, respectively, the mth time a hamburger is added to the stack when we read X backward and the number of cheeseburgers minus the number of cheeseburger orders in $X(-J_m^H, -1)$. Define J_m^C and L_m^C similarly with the roles of hamburgers and cheeseburgers interchanged.

Let

$$\mu := \frac{\pi}{2\left(\pi - \arctan \frac{\sqrt{1-2p}}{p}\right)} = \frac{\kappa}{8} \in (1/2, 1), \quad \mu' := \frac{\pi}{2\left(\pi + \arctan \frac{\sqrt{1-2p}}{p}\right)} = \frac{\kappa}{4(\kappa-2)} \in (1/3, 1/2), \tag{11}$$

where here p and κ are related as in (1). The parameters μ and μ' appear as exponents for several probabilities related to the inventory accumulation model studied in this paper. See [GMS15, BLR15] as well as the later results of this paper.

1.2. **Statements of main results.** Here we will state the main results of this article. The following event will play a key role throughout the paper.

Definition 1.7. For $n, h, c \in \mathbb{N}$, we denote by $\mathcal{E}_n^{h,c}$ the event that $X(1,n)$ contains no orders (i.e. $I > n$), h hamburgers, and c cheeseburgers.

In terms of the path Z^n of (7), $\mathcal{E}_n^{h,c}$ is the event that Z^n stays in the first quadrant for one unit of time and satisfies $Z^n(1) = (n^{-1/2}h, n^{-1/2}c)$. Our first main result is the following scaling limit result for the path Z^n of (7) conditioned on the event $\mathcal{E}_n^{h,c}$.

Theorem 1.8. *Fix $C > 1$. For each $\epsilon > 0$, there exists $n_* \in \mathbb{N}$ such that the following is true. For each $n \geq n_*$ and each $(h,c) \in \left[C^{-1}n^{1/2}, Cn^{1/2}\right]_{\mathbb{Z}}^2$, the Prokhorov distance (with respect to the uniform metric) between the conditional law of Z^n given the event $\mathcal{E}_n^{h,c}$ of Definition 1.7 and the law of a correlated Brownian motion Z as in* (8) *conditioned to stay in the first quadrant for one unit of time and satisfy $Z(1) = (n^{-1/2}h, n^{-1/2}c)$ is at most ϵ.*

See Section 1.3.1 for a precise definition of a Brownian motion under the conditioning of Theorem 1.8. We note that Theorem 1.8 implies in particular that for any sequence of pairs $(h_n, c_n) \in \mathbb{N}^2$ such that $n^{-1/2}h_n \to u > 0$ and $n^{-1/2}c_n \to v > 0$, the conditional law of Z^n given $\mathcal{E}_n^{h_n,c_n}$ converges as $n \to \infty$ to a correlated Brownian motion Z as in (8) conditioned to stay in the first quadrant for one unit of time and satisfy $Z(1) = (u,v)$. Theorem 1.8 extends the scaling limit results [She11, Theorem 2.5] (for the unconditioned law of Z^n) and [GMS15, Theorem A.1] (for the path Z^n conditioned to stay in the first quadrant, but without its location at time 1 specified).

If we could allow $(h,c) = (0,0)$ in Theorem 1.8, we would obtain convergence of the path Z^n in the finite-volume version of Sheffield's bijection, which corresponds to a random planar map on the sphere. However, treating this case will take quite a bit of additional work, both because of the additional conditioning near the tip of the path (it has to stay in the first quadrant despite being very close to the origin) and because difficulties resulting from the presence of flexible orders. The $(h,c) = (0,0)$ case will be treated in the sequel [GS15] to this paper. Theorem 1.8 is in some sense an intermediate step toward a proof of convergence of the path Z^n conditioned on empty reduced word, but we will actually use only the estimates involved in the proof of Theorem 1.8 in [GS15], not the statement of Theorem 1.8 itself.

Theorem 1.8 is also noteworthy in that it gives a scaling limit statement for a non-Markovian random walk conditioned on a very low probability event toward a certain conditioned correlated Brownian motion. This result fits into a large body of literature concerning scaling limits of conditioned random walks. See [Lig68, Bol76, Igl74, Don85, CC13, Soh10, Shi91, Gar11, DW11, GMS15] and the references therein.

Remark 1.9. As a consequence of [GMS15, Theorem 1.9] and Theorem 1.8, one can also obtain an analogue of the cone time convergence statement [GMS15, Theorem 1.9] in the setting of Theorem 1.8. A very similar argument will be given in [GS15] to prove the analogous statement for random planar maps on the sphere, so we do not give the details here.

Our second main result gives the exponent for the probability of the event that a word of length $2n$ sampled according to the probabilities (4) reduces to the empty word.

Theorem 1.10. *For $n \in \mathbb{N}$, we have*

$$\mathbb{P}\left(X(1,2n) = \emptyset\right) = n^{-1-2\mu+o_n(1)},$$

with μ as in (11).

Theorem 1.10 confirms a prediction of Sheffield [She11, Section 4.2] that $\mathbb{P}\left(X(1,2n) = \emptyset\right)$ has polynomial decay, and in fact yields the exact tail exponent. We note that a weaker lower bound, namely that $\mathbb{P}\left(X(1,2n) = \emptyset\right) \succeq n^{-3}$, is obtained in [Che15], but no upper bound is proven in that paper (although a trivial, but far from optimal, polynomial upper bound of $n^{-3/2+o_n(1)}$ follows from the fact that the total number of burgers in $X(1,i)$ minus the total number of orders in $X(1,i)$ evolves as a simple random walk).

In the course of proving Theorems 1.8 and 1.10, we will also prove several "local" estimates for quantities associated with the inventory accumulation model, i.e. estimates for the probability that a certain random word contains a specified number of symbols of a given type. Local estimates like the ones in this paper are necessary for studying finer properties of the model, like questions about the event of Definition 1.7 or the event $\{X(1,2n) = \emptyset\}$. This paper provides a toolbox of local estimates which are applicable whenever one is interested in small scale properties of the word X. Such estimates will be used, for example, in the sequel [GS15] to this work to obtain scaling limit results for the path Z^n of (7) when we condition on $\{X(1,2n) = \emptyset\}$.

1.3. **Preliminaries.**

1.3.1. *Brownian motion conditioned to stay in the first quadrant.* The statement of Theorem 1.8 refers to a correlated Brownian motion as in (8) conditioned to stay in the first quadrant for one unit of time and satisfy $Z(1) = (u, v)$ for some fixed $(u, v) \in (0, \infty)^2$. In this subsection we will describe how to make sense of this object. We first recall how to make sense of a correlated Brownian motion Z conditioned to stay in the first quadrant (see [GMS15, Section 2.1] and [Shi85] for more detail).

In [Shi85], Shimura constructs an uncorrelated two-dimensional Brownian motion conditioned to stay in the cone $\{z \in \mathbb{C} : 0 \leq \arg z \leq \theta\}$ for one unit of time. By choosing $\theta = \theta(p)$ appropriately and applying a linear transformation which takes this cone to the first quadrant, we obtain a path $\widehat{Z}$ which we interpret as the correlated two-dimensional Brownian motion Z in (8) conditioned to stay in the first quadrant for one unit of time. We note that the law of $\widehat{Z}$ is uniquely characterized by the conditions that $\widehat{Z}(t)$ a.s. lies in the interior of the first quadrant at each fixed time $t \in (0, 1)$; and for each $t \in (0, 1)$, the conditional law of $\widehat{Z}|_{[t,1]}$ given $\widehat{Z}|_{[0,t]}$ is that of a Brownian motion with variances and covariances as in (8) started from $\widehat{Z}(t)$ and conditioned on the positive probability event that it stays in the first quadrant for $1 - t$ units of time (see [GMS15, Lemma 2.1]).

Given $(u, v) \in (0, \infty)^2$, the law of a Brownian Z conditioned to stay in the first quadrant for one unit of time and satisfy $Z(1) = (u, v)$ is the regular conditional law of the path $\widehat{Z}$ described above given $\{\widehat{Z}(1) = (u, v)\}$. This law can be sampled from as follows. First fix $t \in (0, 1)$ and sample $Z|_{[0,t]}$ from the law of a Brownian motion with variances and covariances as in (8) conditioned to stay in the first quadrant for t units of time weighted by $f_{1-t}^{Z(t)}(u, v)$, where for $z \in (0, \infty)^2$, f_{1-t}^{z} denotes the density with respect to Lebesgue measure of the law of a Brownian motion as in (8) started from z conditioned to stay in the first quadrant for $1 - t$ units of time. Then, conditioned on $Z|_{[0,t]}$, sample a Brownian bridge from $Z(t)$ to (u, v) in time $1 - t$ conditioned to stay in the first quadrant, and concatenate this Brownian bridge with our given realization of $Z|_{[0,t]}$.

By Brownian scaling, one obtains the conditional law of a Brownian motion as in (8) conditioned to stay in the first quadrant for a general $t > 0$ units of time and satisfy $Z(t) = (u, v) \in (0, \infty)^2$.

The following is the analogue of [GMS15, Lemma 2.1] when we further condition on the terminal point of the path. The proof can be found in [MS15].

Lemma 1.11. *Fix $(u, v) \in (0, \infty)^2$. Let $\widehat{Z}$ have the law of a correlated Brownian motion as in* (8) *conditioned to stay in the first quadrant for one unit of time and end up at (u, v). Then $\widehat{Z}$ satisfies the following two conditions.*

(1) For each fixed $t \in [0, 1]$, a.s. $\widehat{Z}(t) \in (0, \infty)^2$.
(2) For each $t \in (0, 1)$, the regular conditional law of $\widehat{Z}|_{[t,1]}$ given $\widehat{Z}|_{[0,t]}$ is that of a Brownian bridge from $\widehat{Z}(t)$ to (u, v) in time $1 - t$ with variances and covariances as in (8) *conditioned on the (a.s. positive probability) event that it stays in the first quadrant.*

If $\widetilde{Z} : [0, 1] \to [0, \infty)^2$ is another random continuous path satisfying the above two conditions, then $\widetilde{Z} \overset{d}{=} \widehat{Z}$.

1.3.2. *Regular variation.* A function $f : [1, \infty) \to (0, \infty)$ is called *regularly varying of exponent* $\alpha \in \mathbb{R}$ if for each $\lambda > 0$,

$$\lim_{t \to \infty} \frac{f(\lambda t)}{f(t)} = \lambda^{-\alpha}.$$

The function f is called *slowly varying* if f is regularly varying of exponent 0. Every function f which is regularly varying of exponent α can be represented in the form $f(t) = \psi(t) t^{-\alpha}$, where ψ is slowly varying. For each slowly varying function ψ, there exists $t_0 > 0$ and bounded functions $a, b : [0, \infty) \to \mathbb{R}$ with $\lim_{t \to \infty} b(t) = 0$ such that for $t \geq t_0$,

$$\psi(t) = \exp\left(a(t) + \int_{t_0}^{t} \frac{b(s)}{s}\, ds\right).$$

We refer the reader to [BGT87] for more on regularly varying functions.

The following lemmas are proven in [GMS15, Section A.2]. We recall the definition of the exponent μ from (11) and the definition of I from (9).

Lemma 1.12. *The law of I is regularly varying with exponent μ, i.e. there exists a slowly varying function $\psi_0 : [0,\infty) \to (0,\infty)$ such that*

$$\mathbb{P}(I > n) = \psi_0(n) n^{-\mu}, \qquad \forall n \in \mathbb{N}.$$

Lemma 1.13. *Let P be the smallest $j \in \mathbb{N}$ for which $X(-j,-1)$ contains no orders. Then the law of P is regularly varying with exponent $1-\mu$, i.e. there exists a slowly varying function $\psi_1 : [0,\infty) \to (0,\infty)$ such that*

$$\mathbb{P}(P > n) = \psi_1(n) n^{-(1-\mu)}, \qquad \forall n \in \mathbb{N}.$$

We will treat the functions ψ_0 and ψ_1 as fixed throughout this paper.

1.4. **Outline.** The remainder of this article is structured as follows. In Section 2, we will use the bivariate local limit theorem of [Don91] to prove various "local" estimates for the pairs (J_m^H, L_m^H) of (10). The times J_m^H are especially useful because each $X(-J_m^H,-1)$ contains no flexible orders; and because the law of the pair (J_1^H, L_1^H) is in the normal domain of attraction for a bivariate stable law (see Lemma 2.1 below).

In Section 3, we will use the results of Sections 2 to obtain local estimates for the probability that the word $X_1 \dots X_n$ contains no orders and a particular number of burgers of each type. Some of the more technical arguments needed for the proofs in this section are relegated to Appendix A. From these estimates we will deduce Theorem 1.8.

In Section 4, we will use the estimates of Sections 2 and 3 to obtain estimates for the probability that a word has few orders and approximately a given number of burgers of each type. These estimates will be used in the proof of the upper bound in Theorem 1.10 as well as in [GS15].

In Section 5, we will complete the proof of Theorem 1.10. The proof of the lower bound follows from a relatively straightforward argument which is similar to those given in [GMS15, Section 2] (in fact, a version of this argument appeared in a previous version of [GMS15]). The proof of the upper bound is more complicated, and requires the estimates of Section 4 as well as some additional estimates, including a modification of the estimate [GMS15, Lemma 2.8] for the number of flexible orders in a reduced word.

2. Local estimates for times when hamburgers are added

In this section, we will consider a "local limit" type result (i.e. a uniform convergence statement for densities) for the pairs (J_m^H, L_m^H) introduced in (10). This result will turn out to be a straightforward consequence of the bivariate local limit theorem for stable laws proven in [Don91]. We will then prove some refinements on this result in Section 2.2. Most of the estimates of this section will be used in the proofs of Theorems 1.10 and 1.8. However, this section also has the following broader purpose. As we will see in Sections 3 and 4 below, local estimates for the pairs (J_m^H, L_m^H) are the basic tools for the proofs of many other local estimates and scaling limit results for the inventory accumulation model considered in this paper; see also [GS15, GM15] for more applications of these estimates. This section can be read as a general collection of estimates for the pairs (J_m^H, L_m^H) which can be used to establish whatever additional local estimates one might need for Sheffield's inventory accumulation model.

We note that (by symmetry) all of the results of this section are still valid if we instead consider the pairs (J_m^C, L_m^C) where J_m^C is the smallest $j \in \mathbb{N}$ for which $X(-j,-1)$ contains m cheeseburgers and $L_m^H := d\left(X(-J_m^C,-1)\right)$. However, for the sake of brevity we state our results only for the pairs (J_m^H, L_m^H).

2.1. Local limit theorem.

Lemma 2.1. *Define J_m^H and L_m^H for $m \in \mathbb{N}$ as in (10). Also let τ be the smallest $t > 0$ for which $U(-t) = -1$. Then we have the following convergence in law:*

$$\left(m^{-2} J_m^H, m^{-1} L_m^H\right) \to (\tau, V(-\tau)) . \tag{12}$$

Furthermore, there is a constant $a_0 > 0$ such that

$$\mathbb{P}\left(J_1^H > n\right) = (a_0 + o_n(1))n^{-1/2} \tag{13}$$

and there are constants $a_1, a_2 > 0$ such that

$$\mathbb{P}\left(L_1^H > n\right) = (a_1 + o_n(1))n^{-1} \quad \text{and} \quad \mathbb{P}\left(L_1^H \leq -n\right) = (a_2 + o_n(1))n^{-1}.$$

Proof. Suppose we have (using [She11, Theorem 2.5] and the Skorokhod theorem) coupled (Z^n) with Z in such a way that $Z^n \to Z$ uniformly on compacts a.s. Let $\tau_m := m^{-2}J_m^H$. We observe that τ_m is the smallest $t > 0$ for which $U^{m^2}(-t) = 1$. Since Z a.s. crosses the line $\{(x,y) \in \mathbb{R}^2 : x = 1\}$ immediately after hitting this line when run in the reverse direction, it follows that a.s. $\tau_m \to \tau$. By uniform convergence, a.s. $V(-\tau_m) \to V(-\tau)$. By symmetry, we have the analogous statements for τ^* and $U(-\tau^*)$. Thus (12) holds.

The time τ has the law of a stable random variable with index $1/2$ and skewness parameter 1, and (12) implies that J_m^H is in the normal domain of attraction for this law. By the elementary theory of stable processes, we infer that (13) holds.

The process Z is a linear function of a standard two-dimensional Brownian motion. Therefore the law of $V(-\tau)$ are given by an affine transformation of a Cauchy distribution. Since $V^{m^2}(-\tau_m)$ is equal to m^{-1} times the sum of the m iid random variables $d(X(-J_k^H, -J_{k-1}^H - 1))$ for $k \in [1,m]_{\mathbb{Z}}$, we conclude as above that (2.1) holds. □

From Lemma 2.1, we obtain the following statement, which will play a key role in the remainder of this paper.

Proposition 2.2. *Let g be the joint probability density function of the pair $(\tau, V(-\tau))$ from Lemma 2.1. Then we have*

$$\lim_{m\to\infty} \sup_{(k,l)\in\mathbb{N}\times\mathbb{Z}} \left| m^3\mathbb{P}\left((J_m^H, L_m^H) = (k,l)\right) - g\left(\frac{k}{m^2}, \frac{l}{m}\right)\right| = 0. \tag{14}$$

Proof. We observe that the words $X_{-J_m^H} \dots X_{-J_{m-1}^H - 1}$ for $m \in \mathbb{N}$ are iid (where here we set $J_0^H = 0$). Furthermore, since each $X(-J_m^H, -J_{m-1}^H - 1)$ contains no $\boxed{\text{F}}$-symbols, we have

$$L_m^H = \sum_{k=1}^m d^*\left(X(-J_m^H, -J_{m-1}^H - 1)\right).$$

The local limit result (14) now follows from Lemma 2.1 and the local limit theorem for bivariate stable laws [Don91, Theorem 1]. □

When we apply the estimate of Proposition 2.2, it will be convenient to have a tail estimate for the limiting density g.

Lemma 2.3. *Let g be as in Proposition 2.2. Then we have*

$$g(t,v) = a_0 t^{-2}\exp\left(-\frac{a_1 + a_2(v+a_3)^2}{t}\right) \tag{15}$$

for constants $a_0, a_1, a_2, a_3 > 0$ depending only on p. In particular, g is bounded.

Proof. The function g is the joint density of $(\tau, V(\tau))$, where τ is the first time the Brownian motion U in (8) hits -1 and V is the other Brownian motion in (8). Thus marginal density of τ is given by

$$\mathbb{P}\left(t < \tau < t + dt\right) = C_1 t^{-3/2} e^{-\frac{a_1}{t}}$$

for constants $C_1 > 0$ and $a_1 > 0$ depending only on p. We can write $V = \widetilde{V} + a_3 U$, where a_3 is a constant depending on p and $\widetilde{V}$ is a constant times a standard linear Brownian motion independent from U. Therefore, the conditional density of $V(\tau)$ given τ is given by

$$\mathbb{P}\left(v < V(\tau) < v + dv \,|\, \tau \in dt\right) = C_2 t^{-1/2} e^{-\frac{a_2(v+a_3)^2}{t}},$$

for constants $C_2 > 0$ and $a_2 > 0$ depending only on p. Combining these formulae yields the lemma. $\square$

We want to prove an analogue of Lemma 2.1 for (roughly speaking) times at which hamburger orders are added when we run forward. For this we need the following basic fact, whose proof is left to the reader.

Lemma 2.4. *Let (ξ_j) be a sequence of iid non-negative random variables. For $m \in \mathbb{N}$, let $S_m := \sum_{j=1}^m \xi_j$. Let (N_n) be an increasing sequence of positive integer-valued random variables such that N_n/n a.s. converges to a constant $q > 0$. For $n \in \mathbb{N}$, let $\widehat{S}_n := S_{N_n}$. Suppose there is a random variable X and an $\alpha > 0$ such that $n^{-\alpha}\widehat{S}_n \to X$ in law. Then $m^{-\alpha} S_m \to q^{-\alpha} X$ in law.*

Lemma 2.5. *For $m \in \mathbb{N}$, let $\widetilde{I}^H_m$ be the mth smallest $i \in \mathbb{N}$ for which $X(1,i)$ contains no hamburgers. Also let τ be the smallest $t > 0$ for which $U(t) = -1$. Then*

$$m^{-2}\widetilde{I}^H_m \to \left(\frac{4}{1-p}\right)^2 \tau \qquad \textit{in law.} \tag{16}$$

Furthermore, there is a constant $a_0 > 0$ such that

$$\mathbb{P}\left(\widetilde{I}^H_1 > n\right) = (a_0 + o_n(1))n^{-1/2}. \tag{17}$$

Proof. For $m \in \mathbb{N}$, let I^H_m be the smallest $i \in \mathbb{N}$ for which $X(1,i)$ contains at least m hamburger orders. For $m \in \mathbb{N}$, let E_m be the event that $X_{\widetilde{I}^H_m} = \boxed{\mathrm{H}}$ (so that in particular $\widetilde{I}^H_m = \widetilde{I}^H_{m-1} + 1$). Also let $N_m := \sum_{k \le m} \mathbb{1}_{E_k}$. Observe that E_m occurs if and only if $\widetilde{I}^H_m = I^H_k$ for some $k \in \mathbb{N}$. Therefore $\widetilde{I}^H_{N_m} = I^H_m$. Furthermore, the events E_m are independent, and each has probability $(1-p)/4$. By the law of large numbers, we have $N_m/m \to (1-p)/4$ a.s.

Let $\tau_m := m^{-2} I^H_m$. We claim that $\tau_m \to \tau$ in law. Suppose we have (using [She11, Theorem 2.5] and the Skorokhod theorem) coupled (Z^{m^2}) with a correlated Brownian motion Z as in (8) in such a way that $Z^{m^2} \to Z$ uniformly on compacts a.s. Observe that

$$-1 - m^{-1}\mathcal{N}_{\boxed{\mathrm{F}}}\left(X(1, I^H_m)\right) \le U^{m^2}(\tau_m) \le -1. \tag{18}$$

For $\delta > 0$, let τ_δ be the first time $t > 0$ such that $U(t) = -1 + \delta$. Suppose given $\epsilon > 0$. We can find $\delta > 0$ such that with probability at least $1 - \epsilon$, we have $\tau - \tau_\delta \le \epsilon$. By [GMS15, Lemma 2.8], we can find $m_* = m_*(\epsilon) \in \mathbb{N}$ such that for $m \ge m_*$, it holds with probability at least $1 - \epsilon$ that $m^{-1}\mathcal{N}_{\boxed{\mathrm{F}}}\left(X(1, I^H_m)\right) \le \delta$. By (18), for $m \ge m_*$, it holds with probability at least $1 - 2\epsilon$ that $|\tau_m - \tau| \le \epsilon$. Since ϵ is arbitrary, this implies $\tau_m \to \tau$ in probability, hence also in law.

By Lemma 2.4, we have (16). We now obtain (17) in exactly the same manner as in Lemma 2.1. $\square$

Corollary 2.6. *There is a constant $a_0 > 0$ such that for $n \in \mathbb{N}$,*

$$\mathbb{P}\left(n = J^H_m \textit{ for some } m \in \mathbb{N}\right) = (a_0 + o_n(1))n^{-1/2}.$$

Proof. The event that $n = J^H_m$ for some $m \in \mathbb{N}$ is the same as the event that $X(-n+1, -1)$ contains no hamburger orders or flexible orders and $X_{-n} = \mathrm{H}$. By translation invariance this probability is the same as the probability that $X(1,i)$ contains a hamburger for each $i \in [1,n]_{\mathbb{Z}}$, i.e. the probability that $\widetilde{I}^H_1 > n$, with $\widetilde{I}^H_1$ as in Lemma 2.5. The statement of the corollary now follows Lemma 2.5. $\square$

2.2. **Regularity and large deviation estimates.** In this section we will prove two (closely related) types of results, which sharpen the estimates which come from Proposition 2.2: estimates for the probability that $(J^H_m, L^H_m) = (k,l)$ when k and l are far from m^2 and m, respectively; and regularity estimates for the conditional law of $X_1 \dots X_{J^H_m}$ given $\{(J^H_m, L^H_m) = (k,l)\}$.

Lemma 2.7. *For $m \in \mathbb{N}$ and $R > 0$, we have*

$$\mathbb{P}\left(J_m^H \geq R\right) \preceq R^{-1/2} m, \tag{19}$$

$$\mathbb{P}\left(J_m^H \leq R\right) \preceq e^{-am^2/R}, \quad \text{and} \tag{20}$$

$$\mathbb{P}\left(|L_m^H| \geq R\right) \preceq R^{-1} m \tag{21}$$

with $a > 0$ a universal constant and the implicit constants depending only on p.

Proof. Let (Y_m) be an iid sequence of positive stable random variables of index $1/2$. By Lemma 2.1, we can find a constant $A > 0$, depending only on p, and a coupling of two copies (Y_m^1) and (Y_m^2) of the sequence (Y_m) with the sequence (J_m^H) such that $A^{-1} Y_m^1 \leq J_m^H - J_{m-1}^H \leq A Y_m^2$ a.s. for each $m \in \mathbb{N}$. Then with $S_m^i := \sum_{j=1}^m Y_j^i$ for $i \in \{1, 2\}$, we have

$$\mathbb{P}\left(J_m^H \geq R\right) \leq \mathbb{P}\left(S_m^2 \geq (R/A)\right) = \mathbb{P}\left(m^{-2} S_m^2 \geq (R/A) m^{-2}\right)$$

and

$$\mathbb{P}\left(J_m^H \leq R^{-1}\right) \leq \mathbb{P}\left(S_m^1 \leq (A/R)\right) = \mathbb{P}\left(m^{-2} S_m^1 \leq (A/R) m^{-2}\right).$$

By stability, $m^{-2} S_m^i \stackrel{d}{=} Y_1$ for $i \in \{1, 2\}$. Hence we obtain

$$\mathbb{P}\left(J_m^H \geq R\right) \leq \mathbb{P}\left(Y_1 \geq (R/A) m^{-2}\right) \preceq R^{-1/2} m$$

and

$$\mathbb{P}\left(J_m^H \leq R\right) \leq \mathbb{P}\left(Y_1 \leq A R m^{-2}\right) \preceq e^{-am^2/R}$$

for an appropriate constant $a > 0$ as in the statement of the lemma. This yields (19) and (20).

To prove (21), let $(\widetilde{Y}_m)$ be an iid sequence of positive stable random variables of index 1. By Lemma 2.1, we can find a constant $\widetilde{A} > 0$ (depending only on p) and a coupling of two copies $(\widetilde{Y}_m^1)$ and $(\widetilde{Y}_m^2)$ of the sequence $(\widetilde{Y}_m)$ with the sequence (L_m^H) such that a.s.

$$-\widetilde{A} \widetilde{Y}_m^1 \leq \widetilde{L}_m^H - \widetilde{L}_{m-1}^H \leq \widetilde{A} \widetilde{Y}_m^2 \qquad \forall m \in \mathbb{N}.$$

Therefore, with $\widetilde{S}_m^i = \sum_{j=1}^m \widetilde{Y}_j^i$ for $i \in \{1, 2\}$, we have

$$\begin{aligned}
\mathbb{P}\left(|L_m^H| \geq R\right) &\leq \mathbb{P}\left(\widetilde{S}_m^2 \geq (R/\widetilde{a})\right) + \mathbb{P}\left(\widetilde{S}_m^1 \geq (R/\widetilde{a})\right) \\
&\leq 2\mathbb{P}\left(\widetilde{Y}_1 \geq (R/\widetilde{a}) m^{-1}\right) \preceq R^{-1} m.
\end{aligned}$$

□

Lemma 2.8. *Suppose $m \in \mathbb{N}$ and $(k, l) \in \mathbb{N} \times \mathbb{Z}$. Then*

$$\mathbb{P}\left((J_m^H, L_m^H) = (k, l)\right) \preceq m^{-3} \wedge \left(k^{-1/2} m^{-2}\right) \wedge \left(|l|^{-1} m^{-2}\right) \wedge \left(e^{-a_0 m^2/k} m^{-3}\right), \tag{22}$$

with $a_0 > 0$ a universal constant and the implicit constant depending only on p. Furthermore, there is a universal constant $a_1 > 0$ such that for $R > 0$ we have

$$\mathbb{P}\left((J_m^H, L_m^H) = (k, l), \sup_{j \in [1, J_m^H]_{\mathbb{Z}}} |X(-j, -1)| \geq R\right) \preceq \exp\left(-\frac{a_1 m^2}{k} - \frac{a_1 R}{k^{1/2}}\right) m^{-3}, \tag{23}$$

with the implicit constant depending only on p.

Proof. It is clear from Proposition 2.2 and Lemma 2.3 that for any $m \in \mathbb{N}$ and any $(k, l) \in \mathbb{N} \times \mathbb{Z}$, we have

$$\mathbb{P}\left((J_m^H, L_m^H) = (k, l)\right) \preceq m^{-3}, \tag{24}$$

with the implicit constant depending only on p.

To prove the other estimates in the statement of the lemma, let

$$\begin{aligned}
&m' := \lfloor m/2 \rfloor, \quad J_m^{H,1} := J_{m'}^H, \quad J_m^{H,2} := J_m^H - J_{m'}^H, \\
&L_m^{H,1} := L_{m'}^H, \quad \text{and} \quad L_m^{H,2} := d^*\left(X(-J_m^H, -J_{m'}^H - 1)\right).
\end{aligned} \tag{25}$$

Observe that the pairs $(J_m^{H,1}, L_m^{H,1})$ and $(J_m^{H,2}, L_m^{H,2})$ are independent. For $i \in \{1,2\}$, $R > 0$, and $k \in \mathbb{N}$, let

$$\begin{aligned}
E_m^i(R) &:= \left\{ J_m^{H,i} \geq \frac{R}{2} \right\}, \quad F_m^i(R) := \left\{ |L_m^{H,i}| \geq \frac{R}{2} \right\}, \quad G_m^i(R) := \left\{ J_m^{H,i} \leq R \right\} \\
H_m^1(R,k) &:= \left\{ \sup_{j \in [1, J_m^{H,1}]_{\mathbb{Z}}} |X(-j,-1)| \geq \frac{R}{2},\, J_m^{H,1} \leq k \right\} \\
H_m^2(R,k) &:= \left\{ \sup_{j \in [J_m^{H,1}+1, J_m^{H,2}]_{\mathbb{Z}}} |X(-j, -J_m^{H,1} - 1)| \geq \frac{R}{2},\, J_m^{H,2} \leq k \right\}.
\end{aligned}$$

Then we have

$$\begin{gathered}
\left\{ J_m^H \geq R \right\} \subset E_m^1(R) \cup E_m^2(R), \quad \left\{ |L_m^H| \geq R \right\} \subset F_m^1(R) \cup F_m^2(R), \quad \left\{ J_m^H \leq R \right\} \subset G_m^1(R) \cap G_m^2(R) \\
\left\{ \sup_{j \in [1, J_m^H]_{\mathbb{Z}}} |X(-j,-1)| \geq R,\, J_m^H \leq k \right\} \subset H_m^1(R,k) \cup H_m^2(R,k).
\end{gathered}$$

By Lemma 2.7, for $i \in \{1,2\}$ we have

$$\mathbb{P}\left(E_m^i(R)\right) \preceq R^{-1/2} m, \qquad \mathbb{P}\left(F_m^i(R)\right) \preceq R^{-1} m, \qquad \mathbb{P}\left(G_m^i(R)\right) \preceq e^{-a_0 m^2/R},$$

with $a_0 > 0$ a universal positive constant. By [She11, Lemma 3.13], there is a universal constant $a_1' > 0$ such that

$$\mathbb{P}\left(H_m^i(R,k)\right) \preceq e^{-a_1' R / k^{1/2}}. \tag{26}$$

By (24) and independence, for each $i \in \{1,2\}$, the conditional probability that $(J_m^H, L_m^H) = (k,l)$ given any realization of $(J_m^{H,i}, L_m^{H,i})$ is at most a constant (depending only on p) times m^{-3}. Hence

$$\begin{aligned}
\mathbb{P}\left((J_m^H, L_m^H) = (k,l)\right) &\leq \sum_{i=1}^2 \mathbb{P}\left((J_m^H, L_m^H) = (k,l) \,|\, E_m^i(k)\right) \mathbb{P}\left(E_m^i(k)\right) \preceq k^{-1/2} m^{-2} \\
\mathbb{P}\left((J_m^H, L_m^H) = (k,l)\right) &\leq \sum_{i=1}^2 \mathbb{P}\left((J_m^H, L_m^H) = (k,l) \,|\, F_m^i(|l|)\right) \mathbb{P}\left(F_m^i(|l|)\right) \preceq |l|^{-1} m^{-2} \\
\mathbb{P}\left((J_m^H, L_m^H) = (k,l)\right) &\leq \mathbb{P}\left((J_m^H, L_m^H) = (k,l) \,|\, G_m^1(k)\right) \mathbb{P}\left(G_m^1(k)\right) \preceq e^{-a_0 m^2/k} m^{-3}.
\end{aligned}$$

This yields (22). For (23), we first use (26) to get

$$\begin{aligned}
&\mathbb{P}\left((J_m^H, L_m^H) = (k,l),\, \sup_{j \in [1, J_m^H]_{\mathbb{Z}}} |X(-j,-1)| \geq R \right) \\
&\qquad \leq \sum_{i=1}^2 \mathbb{P}\left((J_m^H, L_m^H) = (k,l) \,|\, H_m^{3-i}(R,k)\right) \mathbb{P}\left(H_m^{3-i}(R,k)\right) \\
&\qquad \preceq \sum_{i=1}^2 \mathbb{P}\left((J_m^H, L_m^H) = (k,l) \,|\, H_m^{3-i}(R,k)\right) e^{-a_1' R/k^{1/2}}.
\end{aligned}$$

On the other hand, by the third inequality in (22) applied with either m' or $m - m'$ in place of m, we have

$$\mathbb{P}\left((J_m^H, L_m^H) = (k,l) \,|\, H_m^{3-i}(R,k)\right) \preceq e^{-a_0 m^2/4k} m^{-3}.$$

Combining these estimates yields (23) with $a_1 = a_1' \wedge (a_0/4)$. □

To complement Lemma 2.8, we also have a lower bound for the probability that $(J_m^H, L_m^H) = (k,l)$ and the word $X(-J_m^H, -1)$ has certain unusual behavior.

Lemma 2.9. *Fix $C > 1$ and $\epsilon > 0$. For sufficiently large $m \in \mathbb{N}$ (how large depends only on C and ϵ) and $(k,l) \in \left[C^{-1}m^2, Cm^2\right]_{\mathbb{Z}} \times [-Cm, Cm]_{\mathbb{Z}}$, we have*

$$\mathbb{P}\left((J_m^H, L_m^H) = (k,l),\, \mathcal{N}_{\textcircled{C}}\left(X(-J_m^H, -1)\right) \leq \epsilon m \right) \succeq m^{-3}$$

with the implicit constant depending only on ϵ and C.

Proof. Fix $\delta > 0$ to be chosen later, depending only on ϵ and C, and let $m_\delta := \lfloor (1-\delta)m \rfloor$. Let $E_m^{k,l}(\delta)$ be the event that

$$(J_{m_\delta}^H, L_{m_\delta}^H) \in [k - 2\delta m, k - \delta m]_{\mathbb{Z}} \times [l - \delta m, l + \delta m]_{\mathbb{Z}}$$
$$\text{and}\, \mathcal{N}_{\textcircled{C}}\left(X(-J_{m_\delta}, -1)\right) \leq (\epsilon \wedge \frac{1}{2}C^{-1})m.$$

By [She11, Theorem 2.5] (c.f. the proof Lemma 2.1) we and find $m_* \in \mathbb{N}$, depending only on C, ϵ, and δ such that for $m \geq m_*$,

$$\mathbb{P}\left(E_m^{k,l}(\delta)\right) \succeq 1 \tag{27}$$

with the implicit constant depending only on C, ϵ, and δ. Note that on $E_m^{k,l}(\delta)$ we have $\mathcal{N}_{\boxed{C}}\left(X(-J_{m_\delta}, -1)\right) \geq \frac{1}{2}C^{-1}$. By Lemma 2.10 and independence, if δ is chosen sufficiently small, depending only on C, then we have

$$\mathbb{P}\left((J_m^H, L_m^H) = (k,l),\, \sup_{j \in [1, J_m^H]_{\mathbb{Z}}} |X(-j,-1)| \leq \frac{1}{2}C^{-1} \,|\, E_m^{k,l}(\delta) \right) \succeq m^{-3}$$

with the implicit constant depending only on C, ϵ, and δ. By combining this with (27), we obtain the statement of the lemma. □

From Lemma 2.8, we obtain a regularity estimate when we condition on a particular realization of (J_m^H, L_m^H).

Lemma 2.10. *For each $C > 1$ and each $q \in (0,1)$, there exists $A > 0$ and $m_* = m_* \in \mathbb{N}$, depending only on C and q, such that for $m \geq m_*$ and $(k,l) \in [C^{-1}m^2, Cm^2]_{\mathbb{Z}} \times [-Cm, Cm]_{\mathbb{Z}}$, we have*

$$\mathbb{P}\left(\sup_{j \in [1, J_m^H]} |X(-j,-1)| \leq Am \,\Big|\, (J_m^H, L_m^H) = (k,l) \right) \geq 1 - q.$$

Proof. By (23) of Lemma 2.8, we can find $a > 0$, depending only on C, such that for each $A > 0$, each $m \in \mathbb{N}$, and each $(k,l) \in [C^{-1}m^2, Cm^2]_{\mathbb{Z}} \times [-Cm, Cm]_{\mathbb{Z}}$,

$$\mathbb{P}\left(\sup_{j \in [1, J_m^H]} |X(-j,-1)| \geq Am,\, (J_m^H, L_m^H) = (k,l) \right) \preceq e^{-aA}m^{-3}$$

with the implicit constant depending only on C. By Proposition 2.2, we can find $m_* \in \mathbb{N}$, depending only on C, such that for $m \geq m_*$ and $(k,l) \in [C^{-1}m^2, Cm^2]_{\mathbb{Z}} \times [-Cm, Cm]_{\mathbb{Z}}$, we have

$$\mathbb{P}\left((J_m^H, L_m^H) = (k,l)\right) \succeq m^{-3},$$

with the implicit constant depending only on C. Combining these observations yields the statement of the lemma. □

Finally, we have a regularity estimate for $X_{-n} \dots X_{-1}$ given only that $n = J_m^H$ for some (unspecified) $m \in \mathbb{N}$.

Lemma 2.11. *For $n \in \mathbb{N}$, let E_n be the event that $n = J_m^H$ for some $m \in \mathbb{N}$. For each $q \in (0,1)$, is a constant $A > 0$ depending only on q such that for each $n \in \mathbb{N}$,*

$$\mathbb{P}\left(\sup_{j \in [1,n]_{\mathbb{Z}}} |X(-j,-1)| \leq An^{1/2} \,\Big|\, E_n \right) \geq 1 - q. \tag{28}$$

Proof. By the same argument used to prove (23) of Lemma 2.8, but with only the value of J_m^H (not the value of L_m^H) specified, for each $m \in \mathbb{N}$ and $A > 0$ we have

$$\mathbb{P}\left(\sup_{j\in[1,n]_{\mathbb{Z}}} |X(-j,-1)| > An^{1/2},\, J_m^H = n\right) \preceq e^{-a_0 A n^{1/2}/m} m^{-2}$$

with $a_0 > 0$ a universal constants and the implicit constant depending only on p. Hence for each $C > 0$,

$$\mathbb{P}\left(\sup_{j\in[1,n]_{\mathbb{Z}}} |X(-j,-1)| > An^{1/2},\, E_n\right) \preceq \sum_{m=1}^{\lfloor Cn^{1/2}\rfloor} e^{-a_0 A n^{1/2}/m} m^{-2} + \sum_{m=\lfloor Cn^{1/2}\rfloor}^{n} m^{-2}$$
$$\preceq \int_0^{Cn^{1/2}} e^{-a_0 A n^{1/2}/t} t^{-2}\, dt + C^{-1} n^{-1/2}$$
$$= n^{-1/2} \int_0^{C} e^{-a_0 A s} s^{-2}\, dt + C^{-1} n^{-1/2}.$$

For any given $\alpha > 0$, we can choose C sufficiently large and then A sufficiently large relative to C such that this integral is at most $\alpha n^{-1/2}$. By Lemma 2.6,

$$\mathbb{P}(E_n) \asymp n^{-1/2},$$

with the implicit constant depending only on p. We conclude by choosing α sufficiently small relative to q and dividing. □

3. Local estimates with no orders

3.1. **Setup.** In this subsection, we will use the results of Sections 2 to establish sharp estimates for the probability that the word $X(1,n)$ contains no orders and a specified number of burgers of each type, and for the conditional law of the word $X_1 \dots X_n$ given that this is the case. These estimates will eventually lead to a proof of Theorem 1.8.

Before we commence with the proofs, we describe the notation we will use throughout this section and make some elementary observations about the objects involved.

Recall the definition of the event $\mathcal{E}_n^{h,c}$ from Definition 1.7, the time I from (9), the exponent μ from (11), and the slowly varying function ψ_0 from Lemma 1.12. We also introduce the following additional notation.

Definition 3.1. For a word x, we write

$$\mathfrak{h}(x) := \mathcal{N}_{Ⓗ}(\mathcal{R}(x)), \quad \mathfrak{c}(x) := \mathcal{N}_{Ⓒ}(\mathcal{R}(x)),$$

with $\mathcal{N}_{Ⓗ}$, $\mathcal{N}_{Ⓒ}$, and $\mathcal{R}$ as in Section 1.1.

For $n, m \in \mathbb{N}$, let $K_{n,m}^H$ (resp. $K_{n,m}^C$) be the largest $i \in [1,n]_{\mathbb{Z}}$ for which $\mathcal{N}_{Ⓗ}(X(1,i)) = m$ (resp. $\mathcal{N}_{Ⓒ}(X(1,i)) = m$) and X_{i+1} is a hamburger (resp. cheeseburger) which is not consumed before time n; (or $K_{n,m}^H = 0$ (resp. $K_{n,m}^C = 0$) if no such i exists. On the event $\{I > n\}$, $K_{n,m}^H$ can equivalently be described as the last time $i \in [1,n]_{\mathbb{Z}}$ for which $d(j) \geq m+1$ for each $j \geq i+1$ (or $K_{n,m}^H = 0$ of no such i exists). The time $K_{n,m}^C$ admits a similar description. We write

$$Q_{n,m}^H := \mathcal{N}_{Ⓒ}\left(X(1, K_{n,m}^H)\right) \qquad Q_{n,m}^C := \mathcal{N}_{Ⓒ}\left(X(1, K_{n,m}^C)\right).$$

See Figure 1 for an illustration.

For $r \in \mathbb{N}$, let $J_{n,r}^H$ be the smallest $j \in \mathbb{N}$ for which $X(n-j,n)$ contains r hamburgers and set $L_{n,r}^H := d^*\left(X(n - J_{n,r}^H, n)\right)$. That is, $(J_{n,r}^H, L_{n,r}^H)$ are defined in the same manner as the pairs (10) but with the word read backward from n rather than from -1.

The main idea of the proofs of the above propositions is to condition on a realization of the word X up to time $K_{n,m}^H$ for some $m \in \mathbb{N}$; then read the word backward from time n and apply the results

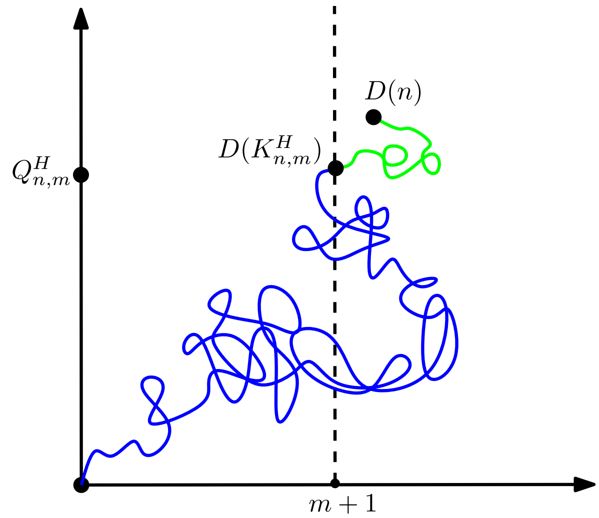

FIGURE 1. An illustration of the time $K^H_{n,m}$, which is -1 plus the last time at which the discrete path D of (6) crosses the vertical line at distance $m+1$ from the origin and subsequently stays to the right of this line. Here $d(n)$, the horizontal coordinate of $D(n)$, is $\geq m$. If $D(n)$ were to the left of the dotted line, then we would have $K^H_{n,m} = 0$. The quantity $Q^H_{n,m}$ is the vertical coordinate of $D(n)$. Note also that $I > n$ in this illustration, i.e. the path D stays in the first quadrant.

of Section 2 to estimate the pairs $(J^H_{n,r}, L^H_{n,r})$. The next two lemmas are the basic tools needed to accomplish this.

Lemma 3.2. *Let D be defined as in (6). For $(h,c) \in \mathbb{N}^2$ and $m \in [1, h-1]_{\mathbb{Z}}$, the event $\mathcal{E}^{h,c}_n$ is the same as the event that*

$$0 < K^H_{n,m} < I, \quad J^H_{n,h-m} = n - 1 - K^H_{n,m}, \quad L^H_{n,h-m} = c - Q^H_{n,m},$$
$$\text{and} \quad \mathcal{N}_{\boxed{C}}\left(X(n - J^H_{n,h-m}, n)\right) \leq Q^H_{n,m}.$$

Proof. It is clear that $K^H_{n,m} > 0$ on $\mathcal{E}^{h,c}_n$. On the event $\{K^H_{n,m} > 0\}$, there is some $j \in \mathbb{N}$ for which $n - J^H_{n,j} = K^H_{n,m} + 1$. Since $X(n - J^H_{n,j}, n)$ contains no hamburger orders or flexible orders, it follows that for this choice of j we have

$$\mathcal{N}_{\text{H}}\left(X(1,n)\right) = m + j, \quad d^*\left(X(1,n)\right) = Q^H_{n,m} + L^H_{n,j},$$
$$\mathcal{N}_{\boxed{\text{C}}}\left(X(1,n)\right) = 0 \vee \left(\mathcal{N}_{\boxed{\text{C}}}\left(X(n - J^H_{n,j}, n)\right) - Q^H_{n,m}\right).$$

The statement of the lemma follows. $\square$

Lemma 3.3. *The marginal law of the pairs $(J^H_{n,r}, L^H_{n,r})$ for $m \in \mathbb{N}$ is the same as the law of the pairs (J^H_r, L^H_r) of Section 2. Furthermore, for each $m \in \mathbb{N}$ the conditional law of $X_{K^H_{n,m}+1} \dots X_n$ given any realization x of $X_1 \dots X_{K^H_{n,m}}$ for which $0 < K^H_{n,m} < I$ is the same as its conditional law given the event*

$$R_n(x) := \{n - 1 - |x| = J^H_{n,r} \text{ for some } r \in \mathbb{N}\}. \tag{29}$$

Proof. The first statement is immediate from translation invariance. To verify this second statement, we observe that for each $k \leq n$, the event $\{K^H_{n,m} = k\} \cap \{K^H_{n,m} < I\}$ is the same as the event that $X(1,k)$ contains no orders and exactly m hamburgers; and that X_{k+1} is a hamburger which does not have a match in $[k+2, n]_{\mathbb{Z}}$, i.e. $k+1 = n - J^H_{n,r}$ for some $r \in \mathbb{N}$. Since $X_1 \dots X_k$ is independent from

$X_{k+1}\dots X_n$, it follows that the conditional law of $X_{k+1}\dots X_n$ given $\{X_1\dots X_{K^H_{n,m}} = x\}$ is the same as its conditional law given that $n-|x|-1 = J^H_{n,r}$ for some $r\in\mathbb{N}$. $\square$

Fix $(h,c)\in\mathbb{N}^2$, $n\in\mathbb{N}$, and $m\in\mathbb{N}$ with $m<h$. Let x be any realization of $X_1\dots X_{K^H_{n,m}}$ for which $0<K^H_{n,m}<I$, so that $\mathfrak{c}(x)$ is the corresponding realization of $Q^H_{n,m}$. Lemmas 3.2 and 3.3 law together yield the following formulae, which we will use frequently in the remainder of this subsection as well as in Appendix A.

$$\begin{aligned}&\mathbb{P}\left(\mathcal{E}^{h,c}_n \,|\, X_1\dots X_{K^H_{n,m}} = x\right)\\ &\qquad = \frac{\mathbb{P}\left((J^H_{n,h-m}, L^H_{n,h-m}) = (n-|x|-1, c-\mathfrak{c}(x)),\, \mathcal{N}_{\boxed{\mathrm{C}}}\left(X(n-J^H_{n,h-m},n)\right)\le \mathfrak{c}(x)\right)}{\mathbb{P}\left(R_n(x)\right)},\end{aligned} \tag{30}$$

with $R_n(x)$ as in (29); and

$$\begin{aligned}&\mathbb{P}\left(\mathcal{E}^{h,c}_n \,|\, X_1\dots X_{K^H_{n,m}} = x,\, I>n\right)\\ &\qquad = \frac{\mathbb{P}\left((J^H_{n,h-m}, L^H_{n,h-m}) = (n-|x|-1, c-\mathfrak{c}(x)),\, \mathcal{N}_{\boxed{\mathrm{C}}}\left(X(n-J^H_{n,h-m},n)\right)\le \mathfrak{c}(x)\right)}{\mathbb{P}\left(R_n(x),\, \mathcal{N}_{\boxed{\mathrm{C}}}\left(X(|x|+1,n)\right)\le\mathfrak{c}(x)\right)}.\end{aligned} \tag{31}$$

3.2. **Upper and lower bounds.** In this subsection we will prove estimates for the probability of the event $\mathcal{E}^{h,c}_n$ of Definition 1.7. We start with the lower bound, which is easier.

Proposition 3.4 (Lower bound). *Let ψ_0 be the slowly varying function from Lemma 1.12. For each $C>1$, $n\ge C^2$, and $(h,c)\in[C^{-1}n^{1/2}, Cn^{1/2}]^2_{\mathbb{Z}}$, we have*

$$\mathbb{P}\left(\mathcal{E}^{h,c}_n \,|\, I>n\right)\succeq n^{-1}$$

with the implicit constant depending only on p.

Proof. Fix $C>1$. For $h\in\mathbb{N}$ and $\delta>0$, let $m^\delta_h := \lfloor(1-\delta)h\rfloor$ and $r^\delta_h := \lfloor\delta h\rfloor$. By Lemma 2.10, we can find $A>0$, depending only on p, such that for each $h\in\mathbb{N}$ and each $(k,l)\in[\frac12\delta^2h^2,\delta^2h^2]_{\mathbb{Z}}\times[-\delta h,\delta h]^2_{\mathbb{Z}}$,

$$\mathbb{P}\left(\sup_{j\in[1,J^H_{r^\delta_h}]_{\mathbb{Z}}} |X(n-j,n)|\le A r^\delta_h,\, (J^H_{r^\delta_h}, L^H_{r^\delta_h}) = (k,l)\right)\succeq \delta^{-3}h^{-3} \tag{32}$$

with the implicit constant depending only on p.

Henceforth fix $\delta\in(0,1/2)$ such that $A\delta\le(2C)^{-1}$, and note that δ depends only on C. By [GMS15, Theorem A.1], for each $n\ge C^2$ and each $(h,c)\in[C^{-1}n^{1/2},Cn^{1/2}]^2_{\mathbb{Z}}$, we have

$$\mathbb{P}\left((K^H_{n,m^\delta_h}, Q^H_{n,m^\delta_h})\in\left[n-\delta^2h^2, n-\frac12\delta^2h^2\right]_{\mathbb{Z}}\times[c-\delta h, c+\delta h]^2_{\mathbb{Z}} \,|\, I>n\right)\succeq 1. \tag{33}$$

By (31), for any realization x of $X_1\dots X_{K^H_{n,m^\delta_h}}$ for which $(K^H_{n,m^\delta_h}, Q^H_{n,m^\delta_h})\in\left[n-\delta^2h^2, n-\frac12\delta^2h^2\right]_{\mathbb{Z}}\times[c-\delta h, c+\delta h]^2_{\mathbb{Z}}$ and $0<K_{n,m^\delta_h}<I$, we have

$$\begin{aligned}&\mathbb{P}\left(\mathcal{E}^{h,c}_n \,|\, X_1\dots X_{K^H_{n,m^\delta_h}} = x,\, I>n\right)\\ &\qquad \ge \frac{\mathbb{P}\left((J^H_{n,r^\delta_h}, L^H_{n,r^\delta_h}) = (n-|x|-1, c-\mathfrak{c}(x)),\, \mathcal{N}_{\boxed{\mathrm{C}}}\left(X(n-J^H_{n,r^\delta_h})\right)\le \mathfrak{c}(x)\right)}{\mathbb{P}\left(R_n(x)\right)},\end{aligned} \tag{34}$$

where $\mathfrak{c}(x) := \mathcal{N}_{\mathrm{C}}(\mathcal{R}(x))$. By (32) and Lemma 2.6, this quantity is

$$\succeq (n-|x|)^{1/2}\delta^{-3}h^{-3}\succeq n^{-1}$$

with the implicit constant depending only on C. By combining this with (33), we obtain the proposition. $\square$

To prove our upper bound for the probability of $\mathcal{E}_n^{h,c}$, we first need the following lemma.

Lemma 3.5. *Let ψ_0 be the slowly varying function from Lemma 1.12. For each $m \in \mathbb{N}$, we have*

$$\mathbb{P}\left(\sup_{i\in[1,I]_{\mathbb{Z}}}\left(\mathcal{N}_{H}(X(1,i)) \wedge \mathcal{N}_{C}(X(1,i))\right) \geq m\right) \preceq \psi_0(m^2)m^{-2\mu}$$

with the implicit constant depending only on p.

Proof. For $m \in \mathbb{N}$, let I_m be the smallest $i \in \mathbb{N}$ for which $\mathcal{N}_{H}(X(1,i)) \wedge \mathcal{N}_{C}(X(1,i) \geq m/2$. Also let $\Bbbk_m$ be the largest $k \in \mathbb{N}$ for which $2^{k-1} \leq m^2$. Observe that if

$$\sup_{i\in[1,I]_{\mathbb{Z}}}\left(\mathcal{N}_{H}(X(1,i)) \wedge \mathcal{N}_{C}(X(1,i))\right) \geq m$$

then $I_m < I$ and $\sup_{i\in[I_m+1,I]_{\mathbb{Z}}} |X(I_m+1,i)| \geq m/2$. Therefore,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{i\in[1,I]_{\mathbb{Z}}}\left(\mathcal{N}_{H}(X(1,i)) \wedge \mathcal{N}_{C}(X(1,i))\right) \geq m\right)\\
&\quad\leq \sum_{k=1}^{\Bbbk_m}\mathbb{P}\left(\sup_{i\in[I_m+1,I_m+2^k]_{\mathbb{Z}}}|X(I_m+1,i)| \geq m/2,\, I-I_m\in[2^{k-1},2^k]_{\mathbb{Z}}\right) + \mathbb{P}\left(I>m^2\right)\\
&\quad\leq \sum_{k=1}^{\Bbbk_m}\mathbb{P}\left(\sup_{i\in[I_m+1,I_m+2^k]_{\mathbb{Z}}}|X(I_m+1,i)| \geq m/2 \,|\, I>I_m+2^{k-1}\right)\mathbb{P}\left(I>2^{k-1}\right) + \mathbb{P}\left(I>m^2\right).
\end{aligned} \tag{35}$$

Let x be any realization of $X_1 \dots X_{I_m}$ for which $I_m < I$. Then for $k \in [1, \Bbbk_m]_{\mathbb{Z}}$,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{i\in[I_m+1,2^k]_{\mathbb{Z}}}|X(I_m+1,i)| \geq m/2 \,\big|\, I>I_m+2^{k-1},\, X_1\dots X_{I_m}=x\right)\\
&\qquad\leq \frac{\mathbb{P}\left(\sup_{i\in[I_m+1,2^k]_{\mathbb{Z}}}|X(I_m+1,i)| \geq m/2 \,|\, X_1\dots X_{I_m}=x\right)}{\mathbb{P}\left(I>I_m+2^{k-1}\,|\,X_1\dots X_{I_m}=x\right)}.
\end{aligned}$$

Since $\mathcal{R}(x)$ contains no orders and at least $m/2$ burgers of each type, and since $2^{k-1} \leq m^2$, it follows from [She11, Theorem 2.5] that $\mathbb{P}\left(I > I_m + 2^{k-1} \,|\, X_1 \dots X_{I_m} = x\right)$ is bounded below by a universal constant. By [She11, Lemma 3.13],

$$\mathbb{P}\left(\sup_{i\in[I_m+1,2^k]_{\mathbb{Z}}}|X(I_m+1,i)| \geq m/2 \,|\, X_1\dots X_{I_m}=x\right) \leq a_0 e^{-a_1 2^{-k/2}m}$$

for universal constants $a_0, a_1 > 0$. By averaging over all realizations of $X_1 \dots X_{I_m}$ for which $I_m < I$, we obtain

$$\mathbb{P}\left(\sup_{i\in[I_m+1,2^k]_{\mathbb{Z}}}|X(I_m+1,i)| \geq m/2 \,\big|\, I>I_m+2^{k-1}\right) \leq a_0 e^{-a_1 2^{-k/2}m}. \tag{36}$$

By Lemma 1.12,

$$\mathbb{P}\left(I > 2^{k-1}\right) \preceq \psi_0(2^k)2^{-k\mu}.$$

Therefore, (35) is at most a constant (depending only on p) times

$$\begin{aligned}\sum_{k=1}^{\Bbbk_m} \psi_0(2^k) 2^{-k\mu} e^{-a_1 2^{-k/2} m} + \psi_0(m^2) m^{-2\mu} \\ &\preceq \psi_0(m^2) m^{-2\mu} \sum_{k=1}^{\Bbbk_m} \frac{\psi_0(2^k)}{\psi_0(m^2)} 2^{(\Bbbk_m - k)\mu} e^{-a_1 2^{(\Bbbk_m - k)/2}} + \psi_0(m^2) m^{-2\mu} \\ &\preceq \psi_0(m^2) m^{-2\mu}.\end{aligned}$$

□

Proposition 3.6 (Upper bound). *Let ψ_0 be the slowly varying function from Lemma 1.12. For each $n \in \mathbb{N}$ and $(h,c) \in \mathbb{N}^2$,*

$$\mathbb{P}\left(\mathcal{E}_n^{h,c}\right) \preceq \psi_0((h \wedge c)^2)(h \wedge c)^{-2-2\mu}$$

with the implicit constant depending only on p.

Proof. Given $(h,c) \in \mathbb{N}^2$, let $m_h = \lfloor h/2 \rfloor$, $r_h = h - m_h$, $m_c = \lfloor c/2 \rfloor$, and $r_c = c - m_c$. Also define K^H_{n,m_h} and K^C_{n,m_c} as in Section 3.1. On the event $\mathcal{E}_n^{h,c}$, both K^H_{n,m_h} and K^C_{n,m_c} are $< n$. Furthermore, if $K^C_{n,m_c} < K^H_{n,m_h}$, then $\mathcal{N}_{\text{C}}\left(X(1, K^H_{n,m_h})\right) \geq m_c$, and the analogous statement holds when $K^H_{n,m_h} \leq K^C_{n,m_c}$. Hence

$$\begin{aligned}\mathbb{P}\left(\mathcal{E}_n^{h,c}\right) &= \mathbb{P}\left(\mathcal{E}_n^{h,c},\, K^C_{n,m_c} < K^H_{n,m_h} < n < I\right) + \mathbb{P}\left(\mathcal{E}_n^{h,c},\, K^H_{n,m_h} < K^C_{n,m_c} < n < I\right) \\ &\leq \mathbb{P}\left(\mathcal{E}_n^{h,c},\, K^H_{n,m_h} < n < I,\, \mathcal{N}_{\text{C}}\left(X(1, K^H_{n,m_h})\right) \geq m_c\right) \\ &\quad + \mathbb{P}\left(\mathcal{E}_n^{h,c},\, K^C_{n,m_c} < n < I,\, \mathcal{N}_{\text{H}}\left(X(1, K^C_{n,m_c})\right) \geq m_h\right) \\ &\leq \mathbb{P}\left(\mathcal{E}_n^{h,c} \,\middle|\, 0 < K^H_{n,m_h} < I,\, \mathcal{N}_{\text{C}}\left(X(1, K^H_{n,m_h})\right) \geq m_c\right) \mathbb{P}\left(0 < K^H_{n,m_h} < I,\, \mathcal{N}_{\text{C}}\left(X(1, K^H_{n,m_h})\right) \geq m_c\right) \\ &\quad + \mathbb{P}\left(\mathcal{E}_n^{h,c} \,\middle|\, 0 < K^C_{n,m_c} < I,\, \mathcal{N}_{\text{H}}\left(X(1, K^C_{n,m_c})\right) \geq m_h\right) \mathbb{P}\left(0 < K^C_{n,m_c} < I,\, \mathcal{N}_{\text{H}}\left(X(1, K^C_{n,m_c})\right) \geq m_h\right).\end{aligned}$$

By symmetry, it suffices to show that

$$\mathbb{P}\left(\mathcal{E}_n^{h,c} \,\middle|\, 0 < K^H_{n,m_h} < I,\, \mathcal{N}_{\text{C}}\left(X(1, K^H_{n,m_h})\right) \geq m_c\right) \preceq (h \wedge c)^{-2}. \tag{37}$$

and

$$\mathbb{P}\left(0 < K^H_{n,m_h} < I,\, \mathcal{N}_{\text{C}}\left(X(1, K^H_{n,m_h})\right) \geq m_c\right) \preceq \psi_0((h \wedge c)^2)(h \wedge c)^{-2\mu}. \tag{38}$$

To this end, fix a realization x of $X_1 \dots X_{K^H_{n,m_h}}$ for which $0 < K^H_{n,m_h} < I$. By (30), we obtain

$$\mathbb{P}\left(\mathcal{E}_n^{h,c} \,\middle|\, X_1 \dots X_{K^H_{n,m_h}} = x\right) \leq \frac{\mathbb{P}\left((J^H_{n,r_h}, L^H_{n,r_h}) = (n - 1 - |x|, c - \mathfrak{c}(x))\right)}{\mathbb{P}\left(R_n(x)\right)}, \tag{39}$$

with J^H_{n,r_h} and L^H_{n,r_h} are as in Section 3.1 and $R_n(x)$ as in (29). By Lemma 2.6,

$$\mathbb{P}\left(R_n(x)\right) \asymp (n - |x|)^{-1/2} \tag{40}$$

with the implicit constant depending only on p. By Lemma 2.8,

$$\mathbb{P}\left((J^H_{n,r_h}, L^H_{n,r_h}) = (n - 1 - |x|, c - \mathfrak{c}(x))\right) \preceq (n - |x|)^{-1/2} h^{-2}$$

with the implicit constant depending only on p. Hence (39) yields

$$\mathbb{P}\left(\mathcal{E}_n^{h,c} \,\middle|\, X_1 \dots X_{K^H_{n,m_h}} = x\right) \preceq h^{-2}.$$

By averaging over all choices of the realization x for which $0 < K^H_{n,h} < I$ and $\mathfrak{c}(x) \geq m_c$, we obtain (37).

For (38), we observe that if $0 < K^H_{n,h} < I$ and $\mathcal{N}_{\text{Ⓒ}}\left(X(1, K^H_{n,h})\right) \geq m_c$, then

$$\sup_{i \in [1,I]_{\mathbb{Z}}} \left(\mathcal{N}_{\text{Ⓗ}}(X(1,i)) \wedge \mathcal{N}_{\text{Ⓒ}}(X(1,i))\right) \geq m_c \wedge m_h.$$

Hence (38) follows from Lemma 3.5. □

3.3. **Regularity estimates.** In this subsection we will consider some regularity results for the conditional law of $X_1 \dots X_n$ given $\mathcal{E}^{h,c}_n$. Our first proposition tells us, roughly speaking, that the pair $(K^H_{n,m}, Q^H_{n,m})$ is unlikely to be too far from (n,c) if m is close to h and we condition on $\mathcal{E}^{h,c}_n$.

Proposition 3.7. *For $n \in \mathbb{N}$, $(h,c) \in \mathbb{N}^2$, $\delta > 0$, and $A > 1$, let*

$$\mathcal{U}^\delta_n(A,h,c) := \left[n - A^2\delta^2 h^2, n - A^{-2}\delta^2 h^2\right]_{\mathbb{Z}} \times \left[c - A\delta h, c + A\delta h\right]_{\mathbb{Z}}. \tag{41}$$

Also write $m^\delta_h := \lfloor (1-\delta)h \rfloor$. For each $C > 1$ and $q \in (0,1)$, there exists $A > 1$ and $\delta_* > 0$ (depending only on C and q) such that the following is true. For each $\delta \in (0, \delta_*]$, there is an $n_* = n_*(\delta, C, q) \in \mathbb{N}$ such that for $n \geq n_*$ and $(h,c) \in \left[C^{-1}n^{1/2}, Cn^{1/2}\right]^2_{\mathbb{Z}}$, we have

$$\mathbb{P}\left((K^H_{n,m^\delta_h}, Q^H_{n,m^\delta_h}) \in \mathcal{U}^\delta_n(A,h,c) \,|\, \mathcal{E}^{h,c}_n\right) \geq 1 - q.$$

The proof of Proposition 3.7 is very technical, so is given in Appendix A to avoid cluttering the present section.

Our next result tells us that if we condition on $\mathcal{E}^{h,c}_n$ and a sufficiently nice realization of $X_1 \dots X_{K^H_{n,m}}$ for m slightly smaller than h, then it is unlikely that $|X(K^H_{n,m}+1, i)|$ is very large for any $i \in [K^H_{n,m}+1, n]_{\mathbb{Z}}$.

Lemma 3.8. *Fix $C > 1$, $A > 0$, and $q \in (0,1)$. Define $\mathcal{U}^\delta_n(A,h,c)$ as in (41) and let $m^\delta_h := \lfloor(1-\delta)h\rfloor$ as in Proposition 3.7. There is a $\delta_* > 0$ and a $B > 0$, depending only on C, A, and q, such that the following is true. For each $n \in \mathbb{N}$, each $\delta \in (0, \delta_*]$, each $(h,c) \in \left[C^{-1}n^{1/2}, Cn^{1/2}\right]^2_{\mathbb{Z}}$, and each realization x of $X_1 \dots X_{K^H_{n,m^\delta_h}}$ for which $(K^H_{n,m^\delta_h}, Q^H_{n,m^\delta_h}) \in \mathcal{U}^\delta_n(A,h,c)$, we have*

$$\mathbb{P}\left(\sup_{i \in [K^H_{n,m^\delta_h}+1, n]_{\mathbb{Z}}} |X(K^H_{n,m^\delta_h}+1, i)| \leq B\delta n^{1/2} \,|\, \mathcal{E}^{h,c}_n,\, X_1 \dots X_{K^H_{n,m^\delta_h}} = x\right) \geq 1 - q. \tag{42}$$

Proof. Let $\delta > 0$, $n \in \mathbb{N}$, $(h,c) \in \left[C^{-1}n^{1/2}, Cn^{1/2}\right]^2_{\mathbb{Z}}$, and let x be a realization of $X_{K_{n,m^\delta_h}+1} \dots X_n$ for which $(K^H_{n,m^\delta_h}, Q^H_{n,m^\delta_h}) \in \mathcal{U}^\delta_n(A,h,c)$. Also let $r^\delta_h := h - m^\delta_h$. By Lemmas 3.2 and 3.3, the conditional law of $X_{K^H_{n,m^\delta_h}+1} \dots X_n$ given $\mathcal{E}^{h,c}_n \cap \{X_1 \dots X_{K^H_{n,m^\delta_h}} = x\}$ is the same as the conditional law of $X_{-J^H_{r^\delta_h}} \dots X_{-1}$ given that $J^H_{r^\delta_h} = n - |x| - 1$, $L^H_{r^\delta_h} = c - \mathfrak{c}(x)$, and $\mathcal{N}_{\boxed{\text{C}}}\left(X(-J^H_{r^\delta_h}, -1)\right) \leq \mathfrak{c}(x)$, where here $(J_{r^\delta_h}, L^H_{r^\delta_h})$ are as in (10). The statement of the lemma now follows from Lemma 2.10. □

3.4. **Continuity estimates.** In this subsection we will prove some lemmas to the effect that the conditional probability of $\mathcal{E}^{h,c}_n$ given a realization of $X_1 \dots X_{K^H_{n,m}}$ for m slightly smaller than h does not depend too strongly on the realization. Throughout this subsection we define the sets $\mathcal{U}^\delta_n(A,h,c)$ as in (41) and let $m^\delta_h := \lfloor(1-\delta)h\rfloor$ be as in Proposition 3.7.

Lemma 3.9. *For each $q \in (0,1)$, $C > 1$, and $A > 0$, there exists $\delta_* > 0$ such that for each $\delta \in (0, \delta_*]$, there exists $\zeta_* > 0$ such that for each $\zeta \in (0, \zeta_*]$, there exists $n_* = n_*(\delta, \zeta, q, C) \in \mathbb{N}$ such that the following is true. Suppose $n \geq n_*$ and $(h,c), (h',c') \in [C^{-1}n^{1/2}, Cn^{1/2}]^2_{\mathbb{Z}}$ with $|(h,c) - (h',c')| \leq \zeta n^{1/2}$. Suppose also that x and x' are realizations of $X_{-K^H_{n,m^\delta_h}} \dots X_{-1}$ and $X_{-K^H_{n,m^\delta_{h'}}} \dots X_{-1}$, respectively, such* *that*

$$(|x|, \mathfrak{c}(x)) \in \mathcal{U}_n^\delta(2A, h, c), \quad (|x'|, \mathfrak{c}(x')) \in \mathcal{U}_n^\delta(2A, h', c'),$$
$$||x| - |x'|| \le \zeta n, \quad \text{and} \quad |\mathfrak{c}(x) - \mathfrak{c}(x')| \le \zeta n^{1/2}. \tag{43}$$

Then we have

$$1 - q \le \frac{\mathbb{P}\left(\mathcal{E}_n^{h,c} \,|\, X_{-K^H_{n,m^\delta_h}} \dots X_{-1} = x,\, I > n\right)}{\mathbb{P}\left(\mathcal{E}_n^{h',c'} \,|\, X_{-K^H_{n,m^\delta_{h'}}} \dots X_{-1} = x',\, I > n\right)} \le \frac{1}{1-q}. \tag{44}$$

Proof of Proposition 3.9. Fix $\alpha > 0$ to be chosen later, depending only on q. Let $r_h^\delta := h - m_h^\delta$. Suppose given a realization x of $X_1 \dots X_{K^H_{n,m^\delta_h}}$ such that $(K^H_{n,m^\delta_h}, Q^H_{n,m^\delta_h}) \in \mathcal{U}_n^\delta(2A, h, c)$ and $0 < K^H_{n,m^\delta_h} < I$. By (31), we have

$$\mathbb{P}\left(\mathcal{E}_n^{h,c} \,|\, X_1 \dots X_{K^H_{n,m_h}} = x,\, I > n\right)$$
$$= \frac{\mathbb{P}\left((J^H_{n,r^\delta_h}, L^H_{n,r^\delta_h}) = (n - |x| - 1, c - \mathfrak{c}(x)),\, \mathcal{N}_{\boxed{\mathrm{C}}}\left(X(n - J^H_{n,r^\delta_h})\right) \le \mathfrak{c}(x)\right)}{\mathbb{P}\left(R_n(x),\, \mathcal{N}_{\boxed{\mathrm{C}}}\left(X(n - J^H_{n,r^\delta_h})\right) \le \mathfrak{c}(x)\right)}.$$

By Lemmas 2.10 and 2.11, we can find $\delta_* > 0$, depending only on α, A, and C, such that for each $\delta \in (0, \delta_*]$ there exists $n_*^0 = n_*^0(\delta, C, A, \alpha) \in \mathbb{N}$ such that for $n \ge n_*^0$, we have

$$\mathbb{P}\left(\mathcal{N}_{\boxed{\mathrm{C}}}\left(X(n - J^H_{n,r^\delta_h})\right) \le \mathfrak{c}(x) \,|\, (J^H_{n,r^\delta_h}, L^H_{n,r^\delta_h}) = (n - |x| - 1, c - \mathfrak{c}(x))\right) \ge 1 - \alpha$$
$$\mathbb{P}\left(\mathcal{N}_{\boxed{\mathrm{C}}}\left(X(n - J^H_{n,r^\delta_h})\right) \le \mathfrak{c}(x) \,|\, R_n(x)\right) \ge 1 - \alpha.$$

By Lemma 2.6, there is a constant $a_0 > 0$, depending only on p, such that for each $\delta \in (0, \delta_*]$ there exists $n_*^1 = n_*^1(\delta, C, A, \alpha) \in \mathbb{N}$ such that for $n \ge n_*^1$ and each realization x as above,

$$a_0(1-\alpha)(n - |x|)^{1/2} \le \mathbb{P}\left(R_n(x)\right) \le \frac{a_0}{1-\alpha}(n - |x|)^{1/2}.$$

Hence if $\delta \in (0, \delta_*]$ and $n \ge n_*^1$,

$$a_0(1-\alpha)^2 \le \frac{\mathbb{P}\left(\mathcal{E}_n^{h,c} \,|\, X_1 \dots X_{K^H_{n,m_h}} = x,\, I > n\right)}{(n-|x|)^{1/2}\mathbb{P}\left((J^H_{n,r^\delta_h}, L^H_{n,r^\delta_h}) = (n - |x| - 1, c - \mathfrak{c}(x))\right)} \le \frac{a_0}{(1-\alpha)^2}. \tag{45}$$

By Proposition 2.2 (in particular, by continuity of the function g of that proposition), we can find $\zeta > 0$, depending only on δ, C, and q, and $n_*^2 = n_*^2(\delta, C, \alpha) \ge n_*^1$ such that for any $n \ge n_*^2$, any two pairs $(h, c), (h', c') \in [C^{-1}n^{1/2}, Cn^{1/2}]^2_{\mathbb{Z}}$ with $|(h, c) - (h', c')| \le \zeta n^{1/2}$, and any realizations x of $X_1 \dots X_{K^H_{n,m^\delta_h}}$ and x' of $X_1 \dots X_{K^H_{n,m^\delta_{h'}}}$ for which (43) holds, we have

$$1 - \alpha \le \frac{\mathbb{P}\left((J^H_{n,r^\delta_h}, L^H_{n,r^\delta_h}) = (n - |x| - 1, c - \mathfrak{c}(x))\right)}{\mathbb{P}\left((J^H_{n,r^\delta_{h'}}, L^H_{n,r^\delta_{h'}}) = (n - |x'| - 1, c' - \mathfrak{c}(x'))\right)} \le \frac{1}{1-\alpha}.$$

By combining this with (45) and choosing α sufficiently small, depending only on q, we obtain the statement of the proposition. ☐

For $n, m, k, l \in \mathbb{N}$ and $\zeta > 0$, define

$$\mathcal{P}^{k,l}_{n,m}(\zeta) := \left\{0 < K^H_{n,m} < I,\, |K^H_{n,m} - k| \le \zeta n,\, |Q^H_{n,m} - l| \le \zeta n^{1/2}\right\}. \tag{46}$$

Lemma 3.10. *For each $q \in (0,1)$, $C > 1$, and $A > 0$, there exists $\delta_* > 0$ such that for each $\delta \in (0, \delta_*]$, there exists $\zeta_* > 0$ such that for each $\zeta \in (0, \zeta_*]$, there exists $n_* = n_*(\delta, \zeta, q, C) \in \mathbb{N}$ such that the following is true. Suppose $n \geq n_*$, $(h,c) \in [C^{-1} n^{1/2}, C n^{1/2}]^2_{\mathbb{Z}}$, and $(k,l) \in \mathcal{U}_n^\delta(A, h, c)$. The conditional law of $X_1 \dots X_{K^H_{n,m_h^\delta}}$ given $\mathcal{P}^{k,l}_{n,m}(\zeta) \cap \mathcal{E}_n^{h,c}$ is mutually absolutely continuous with respect to its conditional law given only $\mathcal{P}^{k,l}_{n,m}(\zeta)$, with Radon-Nikodym derivative bounded above by $(1-q)^{-1}$ and below by $1-q$.*

Proof. Let δ_* be chosen so that the conclusion of Lemma 3.9 holds. Given $\delta \in (0, \delta_*]$, let $\zeta_* > 0$ be chosen as in Lemma 3.9. By possibly shrinking ζ_* we can arrange that for $\zeta \leq \zeta_*$, we have $(k', l') \in \mathcal{U}_n^\delta(A, h, c)$ whenever $(k,l) \in \mathcal{U}_n^\delta(A, h, c)$, $|k - k'| \leq \zeta n$, and $|l - l'| \leq \zeta n^{1/2}$. For $\zeta \in (0, \zeta_*]$, let $n_* = n_*(\delta, \zeta, q, C)$ be as in Lemma 3.9.

Let $n \geq n*$, $(h,c) \in \left[C^{-1} n^{1/2}, C n^{1/2}\right]_{\mathbb{Z}}$, and let x be a realization of $X_{-K_{n,m_h^\delta}} \dots X_{-1}$ for which $\mathcal{P}^{k,l}_{n,m}(\zeta)$ occurs. By applying (44) with $(h,c) = (h', c')$ and averaging over all choices of realization x' for which $\mathcal{P}^{k,l}_{n,m}(\zeta)$ occurs, we obtain

$$1 - q \leq \frac{\mathbb{P}\left(\mathcal{E}_n^{h,c} \,|\, X_{-K^H_{n,m_h^\delta}} \dots X_{-1} = x,\, I > n\right)}{\mathbb{P}\left(\mathcal{E}_n^{h,c} \,|\, \mathcal{P}^{k,l}_{n,m}(\zeta)\right)} \leq \frac{1}{1-q}. \tag{47}$$

By Bayes' rule,

$$\begin{aligned} &\mathbb{P}\left(X_{-K^H_{n,m_h^\delta}} \dots X_{-1} = x \,|\, \mathcal{P}^{k,l}_{n,m}(\zeta) \cap \mathcal{E}_n^{h,c}\right) \\ &\quad = \frac{\mathbb{P}\left(\mathcal{E}_n^{h,c} \,|\, X_{-K^H_{n,m_h^\delta}} \dots X_{-1} = x,\, I > n\right) \mathbb{P}\left(X_{-K^H_{n,m_h^\delta}} \dots X_{-1} = x \,|\, \mathcal{P}^{k,l}_{n,m}(\zeta)\right)}{\mathbb{P}\left(\mathcal{E}_n^{h,c} \,|\, \mathcal{P}^{k,l}_{n,m}(\zeta)\right)}. \end{aligned}$$

By combining this with (47) we conclude. $\square$

3.5. **Proof of Theorem 1.8.** In this subsection we will prove Theorem 1.8. First we need a basic fact about the limiting law. In what follows, let $\widehat{Z} = (\widehat{U}, \widehat{V})$ be a correlated two-dimensional Brownian motion as in (8) conditioned to stay in the first quadrant for one unit of time. For $u > 0$, let τ_u be the last time $t \in [0,1]$ such that $\widehat{U}(t) \geq u$ for each $s \in [t, 1]$; or $\tau_u = 0$ if no such t exists. For $(u,v) \in (0,\infty)^2$ and $t \in [0,1]$, let $\mathbb{P}_t^{u,v}$ be the regular conditional law of $\widehat{Z}$ given $\{\widehat{Z}(t) = (u,v)\}$, as described in Section 1.3.1. Let $\widehat{Z}^{u,v} = (\widehat{U}^{u,v}, \widehat{V}^{u,v})$ be a path distributed according to the law $\mathbb{P}_1^{u,v}$.

Lemma 3.11. *Fix $C > 1$. For each $\epsilon > 0$, there exists $\delta > 0$ such that the following is true. Let $(u,v) \in \left[C^{-1}, C\right]^2$ and $(t, v') \in [1 - \delta^2, 1] \times [v - \delta, v + \delta]$. Also set $u_\delta := (1-\delta) u$. The Prokhorov distance between any two of the following three laws is at most ϵ.*

1. *The regular conditional law of $\widehat{Z}|_{[0,t]}$ given $\{(\tau_{u_\delta}, \widehat{V}(\tau_{u_\delta})) = (t, v')\}$.*
2. *The regular conditional law of $\widehat{Z}^{u,v}|_{[0,t]}$ given $\{(\tau_{u_\delta}, \widehat{V}^{u,v}(\tau_{u_\delta})) = (t, v')\}$.*
3. *The law of $\widehat{Z}^{u,v}|_{[0,t]}$.*

Proof. Since τ_u is the *last* $t \in [0,1]$ for which $\widehat{U}$ crosses u, we infer from the Markov property of $\widehat{Z}$ that for each $s \in (0,t)$, the regular conditional law of $\widehat{Z}|_{[0,s]}$ given $\widehat{Z}|_{[0,s]}$ and $\{(\tau_{u_\delta}, \widehat{V}(\tau_{u_\delta})) = (t, v')\}$ is that of a correlated Brownian bridge from $\widehat{Z}(s)$ to (u_δ, v') in time $t - s$ conditioned to stay in the first quadrant. By Lemma 1.11, the regular conditional law of $\widehat{Z}|_{[0,t]}$ given $\{(\tau_{u_\delta}, \widehat{V}(\tau_{u_\delta})) = (t, v')\}$ is $\mathbb{P}_t^{u_\delta, v'}$. By a similar argument, the regular conditional law of $\widehat{Z}^{u,v}|_{[0,\tau_{u_\delta}]}$ given $\{(\tau_{u_\delta}, \widehat{V}^{u,v}(\tau_{u_\delta})) = (t, v')\}$ is also given by $\mathbb{P}_t^{u_\delta, v'}$. Since the law $\mathbb{P}_t^{u,v}$ depends continuously on its parameters (see Section 1.3.1), it follows that when δ is small and $(t, v') \in [1 - \delta^2, 1] \times [v - \delta, v + \delta]$, the Prokhorov distance between the law of $\widehat{Z}^{u,v}|_{[0,t]}$ and the law $\mathbb{P}_t^{u_\delta, v'}$ is at most ϵ. $\square$

Proof of Theorem 1.8. Fix $\epsilon > 0$ and $C > 1$. By Proposition 3.7 and Lemma 3.8, we can find $A > 1$, $B > 0$, and $\delta_*^0 > 0$ (depending only on ϵ and C) such that the following is true. For each $\delta \in (0, \delta_*^0]$, there exists $n_*^0 = n_*^0(\delta, \epsilon, C) \in \mathbb{N}$ such that for $n \geq n_*^0$ and $(h, c) \in \left[C^{-1} n^{1/2}, C n^{1/2}\right]_{\mathbb{Z}}^2$, we have

$$\mathbb{P}\left((K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta}) \in \mathcal{U}_n^\delta(A, h, c), \sup_{i \in [K^H_{n,m_h^\delta}+1, n]_{\mathbb{Z}}} |X(K^H_{n,m_h^\delta}+1, i)| \leq B\delta n^{1/2} \,|\, \mathcal{E}_n^{h,c} \right) \geq 1 - \epsilon. \tag{48}$$

Choose $\delta_*^1 \in (0, \delta_*^0]$ in such a way that $B\delta_*^0 \leq \epsilon$.

By Lemma 3.10, there exists $\delta_*^2 = \delta_*^2(\epsilon, C) \in (0, \delta_*^0]$ such that for each $\delta \in (0, \delta_*^2]$, there exists $\zeta_* > 0$ such that for each $\zeta \in (0, \zeta_*]$, there exists $n_*^1 = n_*^1(\delta, \zeta, \epsilon, C) \in \mathbb{N}$ such that the following is true. Suppose $n \geq n_*^1$, $(h, c) \in [C^{-1} n^{1/2}, C n^{1/2}]_{\mathbb{Z}}^2$, and $(k, l) \in \mathcal{U}_n^\delta(A, h, c)$. Then the Prokhorov distance between the conditional law of $X_1 \dots X_{K^H_{n,m_h^\delta}}$ given $\mathcal{P}^{k,l}_{n,m_h^\delta}(\zeta) \cap \mathcal{E}_n^{h,c}$ and its conditional law given only $\mathcal{P}^{k,l}_{n,m_h^\delta}(\zeta)$ is at most ϵ, where here $\mathcal{P}^{k,l}_{n,m_h^\delta}(\zeta)$ is as in (46) with $m = m_h^\delta = \lfloor (1-\delta) h \rfloor$.

Define the times τ_u, the laws $\mathbb{P}_t^{u,v}$, and the paths $\widehat{Z}$ and $\widehat{Z}^{u,v}$ as in the discussion just above Lemma 3.11. For $\zeta > 0$, $t \in (0,1)$, and $u, v > 0$, let

$$\mathcal{P}_u^{t,v}(\zeta) := \left\{ |\tau_u - t| \leq \zeta,\, |\widehat{V}(\tau_u) - v| \leq \zeta \right\}. \tag{49}$$

For $\delta > 0$, let $u_\delta := (1-\delta) u$. By averaging the estimate of Lemma 3.11 over pairs $(\widetilde{t}, \widetilde{v}')$ in a small neighborhood of a given pair (t, v'), we can find $\delta = \delta(\epsilon, C) \in (0, \delta_*^2]$ such that the following is true. For each $(u, v) \in \left[C^{-1}, C\right]^2$, each $(t, v') \in [1 - A^2\delta^2, 1] \times [v - 2A\delta, v + 2A\delta]$, and each $\zeta \in (0, \delta]$, the Prokhorov distance between the conditional law of $\widehat{Z}|_{[0,1-A^2\delta^2]}$ given $\mathcal{P}_{u_\delta}^{t,v'}(\zeta)$ and the law of $\widehat{Z}^{u,v}|_{[0,1-A^2\delta^2]}$ is at most ϵ. By possibly decreasing δ and using continuity of the law $\mathbb{P}^{u,v}$ in u and v, we can arrange that for each $(u, v) \in \left[C^{-1}, C\right]^2$, we have

$$\mathbb{P}\left(\sup_{s \in [1 - A^2\delta^2, 1]} |\widehat{Z}^{u,v}(s)| \leq \epsilon \right) \geq 1 - \epsilon. \tag{50}$$

By Lemma A.4, for each $\zeta \in (0, \zeta_* \wedge \delta]$ we can find $n_*^2 = n_*^2(\zeta, \delta, \epsilon, C) \geq n_*^1$ such that for each $n \geq n_*^2$, each $(h, c) \in \left[C^{-1} n^{1/2}, C n^{1/2}\right]_{\mathbb{Z}}$, and each $(k, l) \in \mathcal{U}_n^\delta(A, h, c)$ the Prokhorov distance between the conditional law of $Z^n|_{[0,1]}$ given $\mathcal{P}^{k,l}_{n,m_h^\delta}(\zeta)$ and the conditional law of $\widehat{Z}|_{[0,1]}$ given $\mathcal{P}^{k/n, l/n^{1/2}}_{(1-\delta)h/n^{1/2}}(\zeta)$ is at most ϵ. By our choices of parameters above, we also have that the Prokhorov distance between the conditional law of $Z^n|_{[0,1-A^2\delta^2]}$ given $\mathcal{P}^{k,l}_{n,m_h^\delta}(\zeta) \cap \mathcal{E}_n^{h,c}$ and the law of $\widehat{Z}^{h/n^{1/2}, c/n^{1/2}}|_{[0,1-A^2\delta^2]}$ is at most 3ϵ.

Since this holds for each choice of $(k, l) \in \mathcal{U}_n^\delta(A, h, c)$ and by our choice of A, it follows that the Prokohorov distance between the conditional law of $Z^n|_{[0,1-A^2\delta^2]}$ given $\mathcal{E}_n^{h,c}$ and the conditional law of $\widehat{Z}^{h/n^{1/2}, c/n^{1/2}}|_{[0,t-A^2\delta^2]}$ is at most 4ϵ. By (48), our choice of δ_*^1, and (50), it follows that the Prokhorov distance between the conditional law of $Z^n|_{[0,1]}$ given $\mathcal{E}_n^{h,c}$ and the conditional law of $\widehat{Z}^{h/n^{1/2}, c/n^{1/2}}|_{[0,1]}$ is at most 8ϵ. Since ϵ is arbitrary we conclude. □

4. Local estimates with few orders

In the previous section, we proved various estimates associated with the event that a word contains no orders and a particular number of burgers of each type. In this subsection we will use the results of the previous subsections to prove analogous estimates for the event that a word contains a small number of orders and approximately a particular number of burgers of each type. The estimates of this section are needed for the proof of the upper bound in Theorem 1.10, and will also be used in [GS15].

We will start in Section 4.1 by proving a regular variation type estimate for the probability that there is a time j in a given discrete interval with the property that $X(-j, -1)$ contains no orders. In Section 4.2 we will state and prove our local estimates.

4.1. **Probability of a time with no orders in a small interval.** In this section we will prove a sharp estimate for the probability that there is a time slightly smaller than n for which $X(-j,-1)$ contains no orders. We recall the definitions of regular varying and slowly varying from Section 1.3.2.

Lemma 4.1. *Let ψ_0 be the slowly varying function from Lemma 1.12. There is a slowly varying function ψ_2 such that for $n \in \mathbb{N}$ and $k \in [1,n]_{\mathbb{Z}}$, we have*

$$\mathbb{P}\left(\exists\, j \in [n-k,n]_{\mathbb{Z}} \text{ such that } X(-j,-1) \text{ contains no orders}\right) = (1+o_{k/n}(1))\psi_0(n)\psi_2(k)(k/n)^{\mu},$$

with μ as in (11).

To prove Lemma 4.3 we need some basic facts about regularly varying functions. We leave the proof of the first fact to the reader.

Lemma 4.2. *Let $g : (0,\infty) \to (0,\infty)$ be bounded and regularly varying with exponent $\alpha \in (0,1)$. For $t > 0$, let $\widetilde{g}(t) := \int_0^{\lfloor t \rfloor} g(s)\, ds$. Then $\widetilde{g}$ is regularly varying with exponent $-(1-\alpha)$.*

Lemma 4.3. *Let (Y_j) be an iid sequence of non-negative random variables. For $m \in \mathbb{N}$, let $S_m := \sum_{j=1}^m Y_j$. For $n \in \mathbb{N}$ and $k \in [1,n]_{\mathbb{Z}}$, let G_n^k be the event that there is an $m \in \mathbb{N}$ with $S_m \in [n-k,n]_{\mathbb{Z}}$. Assume that the functions $t \mapsto \mathbb{P}(Y_1 \geq t)$ and $t \mapsto \mathbb{P}\left(\lfloor t \rfloor = S_m \text{ for some } m \in \mathbb{N}\right)$ are regularly varying with exponents α and $1-\alpha$, respectively, so that in particular $\mathbb{P}\left(n = S_m \text{ for some } m \in \mathbb{N}\right) = \psi(n) n^{-(1-\alpha)}$ for some slowly varying function ψ. There is a slowly varying function $\widetilde{\psi}$, depending only on the law of Y_1, such that*

$$\mathbb{P}\left(G_n^k\right) = (1+o_{k/n}(1))\psi(n)\widetilde{\psi}(k)(k/n)^{1-\alpha}.$$

Proof. For $n \in \mathbb{N}$ and $k \in [1,n]_{\mathbb{Z}}$, let M_n^k be the largest $m \in \mathbb{N}$ for which $S_m \in [n-k,n]_{\mathbb{Z}}$ if such a time exists and $M_n^k := n+1$ otherwise. Then we have

$$\mathbb{P}\left(G_n^k\right) = \sum_{i=n-k}^{n} \mathbb{P}\left(S_{M_n^k} = i\right).$$

For $i \in [n-k,n]_{\mathbb{Z}}$, the event $\{S_{M_n^k} = i\}$ is the intersection of the event that $i = S_m$ for some $m \in \mathbb{N}$ and the event that for this m, we have $Y_{m+1} \geq n-i+1$. By the Markov property, the conditional probability of the latter event given the former is the same as the probability that $Y_1 \geq n-j+1$. Thus

$$\mathbb{P}\left(S_{M_n^k} = i\right) = f(i)g(n-i+1)$$

where $f(i) := \mathbb{P}\left(i = S_m \text{ for some } m \in \mathbb{N}\right)$ and $g(i) := \mathbb{P}\left(Y_1 \geq i\right)$. Since $f(n) = \psi(n)n^{-(1-\alpha)}$, we have

$$\mathbb{P}\left(G_n\right) = \sum_{i=n-k}^{n} f(i)g(n-i+1) = (1+o_{k/n}(1))f(n)\sum_{i=n-k}^{n} g(n-i+1) = (1+o_{k/n}(1))\psi(n)n^{-(1-\alpha)}\sum_{j=1}^{k} g(j).$$

By Lemma 4.2, $t \mapsto \sum_{j=1}^{\lfloor t \rfloor} g(j)$ is regularly varying of exponent $-(1-\alpha)$, so there is a slowly varying function $\widetilde{\psi}$ such that

$$\sum_{j=1}^{k} g(j) = \widetilde{\psi}(k)k^{1-\alpha}.$$

The statement of the lemma follows. □

Proof of Lemma 4.1. Let $P_0 = 0$ and for $m \in \mathbb{N}$, let P_m be the mth smallest $j \in \mathbb{N}$ for which $X(-j,-1)$ contains no orders. Then the increments $P_m - P_{m-1}$ are iid, and by [GMS15, Lemma A.8], P_1 is regularly varying with exponent $1-\mu$. By translation invariance,

$$\mathbb{P}\left(\exists\, m \in \mathbb{N} \text{ such that } P_m = n\right) = \mathbb{P}\left(I > n\right).$$

By Lemma 1.12, this quantity is regularly varying in n with exponent μ and slowly varying correction ψ_0. The statement of the lemma now follows from Lemma 4.3. □

4.2. **Statement and proof of the estimates.** In this subsection we will use the following notation. For $n,k\in\mathbb N$ and $(h,c)\in\mathbb N^2$, let $E^{h,c}_{n,k,r}$ be the event that there exists $j\in\mathbb N$ such that the following is true:

$$
\begin{gathered}
j\in[n-k,n]_{\mathbb Z},\quad \mathcal N_{\text{H}}(X(-j,-1))\in[h-r,h]_{\mathbb Z}\,,\quad \mathcal N_{\text{C}}(X(-j,-1))\in[c-r,c]_{\mathbb Z}\,,\\
\mathcal N_{\boxed{\text{H}}}(X(-j,-1))+\mathcal N_{\boxed{\text{F}}}(X(-j,-1))\le r,\quad\text{and}\quad \mathcal N_{\boxed{\text{C}}}(X(-j,-1))+\mathcal N_{\boxed{\text{F}}}(X(-j,-1))\le r.
\end{gathered}\tag{51}
$$

Let $J^{h,c}_{n,k,r}$ be the minimum of $n+1$ and the smallest $j\in[n-k,n]_{\mathbb Z}$ for which (51) holds.

For $m\in\mathbb N$, let J^H_m and L^H_m be as in (10). Let $\widetilde E^{h,c}_{n,k,r}$ be the event that the following is true:

$$
J^H_h\in[n-k,n]_{\mathbb Z}\,,\quad L^H_h\in[c-r,c]_{\mathbb Z}\,,\quad\text{and}\quad \mathcal N_{\text{C}}\left(X(-J^H_h,-1)\right)\le c.\tag{52}
$$

We note that $L^H_h=\mathcal N_{\text{C}}\left(X(-J^H_h,-1)\right)-\mathcal N_{\boxed{\text{C}}}\left(X(-J^H_h,-1)\right)$, so if $\widetilde E^{h,c}_{n,k,r}$ occurs then

$$
\mathcal N_{\boxed{\text{C}}}\left(X(-J^H_h,-1)\right)\le r.\tag{53}
$$

Since $X(-J^H_h,-1)$ contains no hamburger orders or flexible orders, it follows that

$$
\widetilde E^{h,c}_{n,k,r}\subset E^{h,c}_{n,k,r}.
$$

The main result of this section is the following proposition, which is an analogue of Propositions 3.4 and 3.6 of Section 3 with the event $E^{h,c}_{n,k,r}$ or $\widetilde E^{h,c}_{n,k,r}$ in place of the event $\mathcal E^{h,c}_n$. We also include a regularity estimate.

Proposition 4.4. *Suppose we are in the setting described just above. Let ψ_0 be the slowly varying function of Lemma 1.12 and let ψ_2 be the slowly varying function of Lemma 4.1. Fix $C>1$. Suppose $n,k,r,h,c\in\mathbb N$ with $k\le n$, $C^{-1}k\le r^2\le Ck$, and $r\le C(h\wedge c)$.*

(1) For each n,h,c,r,k as above, we have

$$
\mathbb P\left(E^{h,c}_{n,k,r}\right)\preceq\psi_0((h\wedge c)^2)\psi_2(r^2)(h\wedge c)^{-2-2\mu}k^{1+\mu}
$$

with the implicit constants depending only on C.

(2) There exists $m_*\in\mathbb N$, depending only on C, such that for each n,h,c,r,k as above with $n,h,c,r,k\ge m_*$ and $(h,c)\in\left[C^{-1}n^{1/2},Cn^{1/2}\right]^2_{\mathbb Z}$, we have

$$
\mathbb P\left(\widetilde E^{h,c}_{n,k,r}\right)\succeq\psi_0(n)\psi_2(k)n^{-1-\mu}k^{1+\mu}
$$

with the implicit constants depending only on C.

(3) For each $q\in(0,1)$, there exists $A>0$ and $m_*\in\mathbb N$ (depending only on C and q) such that for each n,h,c,r,k as above with $n,h,c,r,k\ge m_*$ and $(h,c)\in\left[C^{-1}n^{1/2},Cn^{1/2}\right]^2_{\mathbb Z}$, we have

$$
\mathbb P\left(\sup_{j\in\left[1,J^{h,c}_{n,k,r}\right]_{\mathbb Z}}|X(-j,-1)|\le An^{1/2}\,|\,E^{h,c}_{n,k,r}\right)\ge1-q.
$$

Remark 4.5. Only assertion 1 of Proposition 4.4 is needed for the proof of Theorem 1.10. However, the other assertions do not take much additional effort to prove and will be used in [GS15].

We will extract Proposition 4.4 from the following lemma, which in turn is a consequence of the results of Section 3 together with Lemma 4.1.

Lemma 4.6. *For $n,k\in\mathbb N$, let $\overline P_{n,k}$ be the largest $j\in[n-k,n]_{\mathbb Z}$ for which $X(-j,-1)$ contains no orders (or $\overline P_{n,k}=0$ if no such j exists). For $n,k,r\in\mathbb N$ and $(h,c)\in\mathbb N^2$, let $\overline E^{h,c}_{n,k,r}$ be the event that $\overline P_{n,k}>0$ and*

$$
\left|\mathcal N_{H}\left(X(-\overline P_{n,k},-1)\right)-h\right|\le r,\qquad \left|\mathcal N_{C}\left(X(-\overline P_{n,k},-1)\right)-c\right|\le r.
$$

Let ψ_0 be the slowly varying function of Lemma 1.12 and let ψ_2 be the slowly varying function of Lemma 4.1. Fix $C > 1$. Suppose $n \in \mathbb{N}$, $(h,c) \in \mathbb{N}^2$ with $h, c \geq n^{\xi/2}$, $k \in [1, n/2]_{\mathbb{Z}}$, and $r \in [1, C(h \wedge c)]_{\mathbb{Z}}$.

(1) For each n, h, c, r, k as above, we have

$$\mathbb{P}\left(\overline{E}^{h,c}_{n,k,r}\right) \preceq \psi_0((h \wedge c)^2)\psi_2(k)(h \wedge c)^{-2-2\mu} r^2 k^\mu$$

with the implicit constants depending only on C.

(2) There exists $n_* \in \mathbb{N}$ (depending only on C) such that for each $n \geq n_*$ and each n, h, c, r, k as above with $(h,c) \in \left[C^{-1} n^{1/2}, C n^{1/2}\right]^2_{\mathbb{Z}}$, we have

$$\mathbb{P}\left(\overline{E}^{h,c}_{n,k,r}\right) \succeq \psi_0(n)\psi_2(k) n^{-1-\mu} r^2 k^\mu$$

with the implicit constants depending only on C.

(3) For each $q \in (0,1)$, there exists $A > 0$ and $n_* \in \mathbb{N}$ (depending only on C and q) such that for each n, h, c, r, k as above with $n \geq n_*$ and $(h,c) \in \left[C^{-1} n^{1/2}, C n^{1/2}\right]^2_{\mathbb{Z}}$, we have

$$\mathbb{P}\left(\sup_{j \in [1, \overline{P}_{n,k}]_{\mathbb{Z}}} |X(-j,-1)| \leq A n^{1/2} \,|\, \overline{E}^{h,c}_{n,k,r} \right) \geq 1 - q.$$

Proof. Observe that

$$\{\overline{P}_{n,k} < \infty\} = \{\exists\, j \in [n-k, n]_{\mathbb{Z}} \text{ such that } X(-j,-1) \text{ contains no orders}\}.$$

By Lemma 4.1,

$$\mathbb{P}\left(\overline{P}_{n,k} > 0\right) \asymp \psi_0(n)\psi_2(k)(k/n)^\mu \tag{54}$$

with ψ_0 as in Lemma 1.12, ψ_2 as in Lemma 4.1, and the implicit constants depending only on p.

For $m \in [n-k, n]_{\mathbb{Z}}$, the event $\{\overline{P}_{n,k} = m\}$ is the same as the event that $X(-m,-1)$ contains no orders and there is no $j \in [m+1, n]$ for which $X(-j, -m+1)$ contains no orders. Hence the conditional law of $X_{-m} \dots X_{-1}$ given $\{\overline{P}_{n,k} = m\}$ is the same as its conditional law given that $X(-m,-1)$ contains no orders. By Lemma 1.12 and Proposition 3.6, it follows that for each $m \in [n-k, n]_{\mathbb{Z}}$,

$$\mathbb{P}\left(\mathcal{E}^{h,c}_m \,|\, \overline{P}_{n,k} = m\right) \preceq \frac{\psi_0((h \wedge c)^2)(h \wedge c)^{-2-2\mu}}{\psi_0(n) n^{-\mu}} \tag{55}$$

with the implicit constant depending only on p. By Proposition 3.4, for each $C > 1$ there exists $n_* \in \mathbb{N}$ such that for each $n \geq n_*$ and each $(h,c) \in \left[(2C)^{-1} n^{1/2}, 2C n^{1/2}\right]^2_{\mathbb{Z}}$,

$$\mathbb{P}\left(\mathcal{E}^{h,c}_m \,|\, \overline{P}_{n,k} = m\right) \succeq n^{-1} \tag{56}$$

with the implicit constant depending only on C. By Theorem 1.8, for each $C > 1$ and $q \in (0,1)$, there exists $A > 0$ and $n_* \in \mathbb{N}$ (depending only on C and q) such that for each $n \in \mathbb{N}$, each $m \in [n-k, n]_{\mathbb{Z}}$, and each $(h,c) \in \left[(2C)^{-1} n^{1/2}, 2C n^{1/2}\right]_{\mathbb{Z}} \times^2$, we have

$$\mathbb{P}\left(\sup_{j \in [1, m]_{\mathbb{Z}}} |X(-j,-1)| \leq A n^{1/2} \,|\, \mathcal{E}^{h,c}_m,\, \overline{P}_{n,k} = m \right) \geq 1 - q. \tag{57}$$

We obtain assertion 1 by combining (54) and (55) then summing over all $(h', c') \in [0 \vee (h-r), h+r]_{\mathbb{Z}} \times [0 \vee (c-r), c+r]_{\mathbb{Z}}$. We similarly obtain assertion 2 from (54) and (56). Assertion 3 is immediate from (57). □

The following lemma will be used to deduce Proposition 4.4 from Lemma 4.6.

Lemma 4.7. *For $n, k, r, h, c \in \mathbb{N}$, let $\overline{P}_{n,k}$ and $\overline{E}^{h,c}_{n,k,r}$ be defined as in Lemma 4.6. Let $E^{h,c}_{n,k,r}$ be defined as in (51) and let $J^{h,c}_{n,k,r}$ be as in the discussion just after. For each $C > 1$ and each $q \in (0,1)$, there is an $m_* \in \mathbb{N}$ and a $B > 0$, depending only on C and q, such that for each $n, h, c, r, k \in \mathbb{N}$ with $k \leq n$,*

$C^{-1}k \leq r^2 \leq Ck$, *and* $r \leq C(h \wedge c)$ *and each realization* x *of* $X_{-J^{h,c}_{n,k,r}} \dots X_{-1}$ *for which* $E^{h,c}_{n,k,r}$ *occurs, we have*

$$\mathbb{P}\left(\overline{E}^{h,c}_{n+Br^2,Br^2-k,Br} \,|\, X_{-J^{h,c}_{n,k,r}} \dots X_{-1} = x\right) \geq 1-q.$$

Proof. Given n,h,c,r,k and a realization x as in the statement of the lemma, let $F(x;B) = F^{h,c}_{n,k,r}(x;B)$ be the event that there is a $j \in [|x|+k, |x|+Br^2]_{\mathbb{Z}}$ such that $X(-j,-|x|-1)$ contains no orders and between r and Br burgers of each type. Then we have

$$\{X_{-J^{h,c}_{n,k,r}} \dots X_{-1} = x\} \cap F(x;B) \subset \overline{E}^{h,c}_{n+Br^2,Br^2,Br},$$

so we just need to show

$$\mathbb{P}\left(F(x;B) \,|\, X_{-J^{h,c}_{n,k,r}} \dots X_{-1} = x\right) \geq 1-q. \tag{58}$$

By [She11, Theorem 2.5], we can find $B_0 > C^2\,(\geq k/r^2)$ such that for each $r>0$, it holds with probability at least $1-q/3$ that the word $X(-|x|-B_0r^2, -|x|-1)$ contains at least r burgers of each type. By Lemma 1.13 and the Dynkin-Lamperti theorem [Dyn55, Lam62], we can find $B_1 > B_0$, depending only on q, such that with probability at least $1-q/3$, there is a $j \in [|x|+B_0r^2, |x|+B_1r^2]_{\mathbb{Z}}$ such that $X(-j,-|x|-1)$ contains no orders. By a second application of [She11, Theorem 2.5], we can find $B_2 > 0$ such that with probability at least $1-q/3$, we have $\sup_{j\in[|x|+1,|x|+B_1r^2]_{\mathbb{Z}}} |X(-j,-|x|-1)| \leq B_2 r$. Since $J^{h,c}_{n,k,r}$ is a stopping time for X, read backward, the word $\dots X_{-|x|-2}X_{-|x|-1}$ is independent from the event $\{X_{-J^{h,c}_{n,k,r}} \dots X_{-1} = x\}$. By combining these observations, we obtain (58) with $B = B_0 \vee B_1 \vee (B_2+1)$. □

Proof of Proposition 4.4. Suppose given n,k,r,h,c as in the statement of the proposition and define the event $\overline{E}^{h,c}_{n,k,r}$ as in Lemma 4.6. By Lemma 4.7, we can find $B>0$, depending only on C, such that

$$\mathbb{P}\left(\overline{E}^{h,c}_{n+Br^2,Br^2-k,Br} \,|\, E^{h,c}_{n,k,r}\right) \geq \frac{1}{2}.$$

By combining this with assertion 1 of Lemma 4.6, we obtain assertion 1 of the present proposition.

From now on we assume further that that $(h,c) \in \left[C^{-1}n^{1/2}, Cn^{1/2}\right]_{\mathbb{Z}}$. To obtain assertion (2), let $\underline{P}^{h,c}_{n,k,r}$ be the smallest $j \in [n-k,n]_{\mathbb{Z}}$ for which

$$\left|\mathcal{N}_{\text{H}}(X(-j,-1)) - h\right| \leq r \quad \text{and} \quad \left|\mathcal{N}_{\text{C}}(X(-j,-1)) - c\right| \leq r,$$

or $\underline{P}^{h,c}_{n,k,r} = n+1$ if no such j exists. Observe that $\underline{P}^{h,c}_{n,k,r}$ is a stopping time for the filtration generated by X, read backward; and $\underline{P}^{h,c}_{n,k,r} \leq n$ on $\overline{E}^{h,c}_{n,k,r}$. By assertion 2 of Lemma 4.6, if n is chosen sufficiently large (depending only on C) then

$$\mathbb{P}\left(\underline{P}^{h,c}_{n,k,r} \leq n\right) \succeq \psi_0(n)\psi_2(k) r^2 k^{\mu} n^{-1-2\mu} \tag{59}$$

with the implicit constant depending only on C. Set $\underline{P}^* := \underline{P}^{h-r/8,c-r/8}_{n-r^2,r/8}$. By [She11, Theorem 2.5] (and since $C^{-1}r^2 \leq k \leq Cr^2$) we can find $m_* \in \mathbb{N}$, depending only on C, such that for n,h,c,k,r as above with $n,h,c,k,r \geq m_*$,

$$\mathbb{P}\left(\widetilde{E}^{h,c}_{n,k,r}(x) \,|\, \underline{P}^* \leq n-r^2\right) \succeq 1,$$

implicit constant depending only on C. By (59) (applied with slightly perturbed values of n, r, h, and c) we obtain assertion 2.

It remains to prove assertion 3. For $A>0$, let

$$G(A) = G^{h,c}_{n,k,r}(A) := E^{h,c}_{n,k,r} \cap \left\{ \sup_{j\in\left[1,J^{h,c}_{n,k,r}\right]_{\mathbb{Z}}} |X(-j,-1)| \leq An^{1/2} \right\}.$$

By Lemma 4.7, we can find a $B > 1$, depending only on C, such that for each n, h, c, k, r as above, we have

$$\mathbb{P}\left(\overline{E} \,|\, G(A)^c \cap E^{h,c}_{n,k,r}\right) \geq \frac{1}{2},$$

where $\overline{E} := \overline{E}_{n+Br^2, Br^2-k, Br}$. By assertions 1 and 2 of Lemma 4.6 together with assertions 1 and 2 of the present lemma, there is an $\widetilde{m}_* \in \mathbb{N}$, depending only on C, such that for n, h, c, r, k as above with $n, h, c, r, k \geq \widetilde{m}_*$, we have

$$\mathbb{P}\left(\overline{E}\right) \asymp \mathbb{P}\left(E^{h,c}_{n,k,r}\right)$$

with the implicit constant depending only on C.

By assertion 3 of Lemma 4.6, for each $\alpha > 0$ we can find $A > 0$ and $m_* \geq \widetilde{m}_*$, depending only on C and α, such that for $m \geq m_*$, we have

$$\mathbb{P}\left(\sup_{j \in [1,n]_{\mathbb{Z}}} |X(-j,-1)| > A n^{1/2} \,|\, \overline{E}\right) \leq \alpha,$$

which implies

$$\mathbb{P}\left(G(A)^c \cap E^{h,c}_{n,k,r} \,|\, \overline{E}\right) \leq \alpha.$$

Hence

$$\mathbb{P}\left(G(A)^c \,|\, E^{h,c}_{n,k,r}\right) = \frac{\mathbb{P}\left(G(A)^c \cap E^{h,c}_{n,k,r} \,|\, \overline{E}\right)\mathbb{P}\left(\overline{E}\right)}{\mathbb{P}\left(\overline{E} \,|\, G(A)^c \cap E^{h,c}_{n,k,r}\right)\mathbb{P}\left(E^{h,c}_{n,k,r}\right)} \preceq \alpha$$

with the implicit constant depending only on C. We now conclude by choosing α smaller than q divided by this implicit constant. □

5. Empty reduced word exponent

In this section we will prove Theorem 1.10. The proof of the lower bound for $\mathbb{P}(X(1,2n) = \emptyset)$ is easier, and is given in Section 5.1. The argument is similar to some of the proofs given in [GMS15, Section 2] (and in fact appeared in an earlier version of that paper). In Section 5.2, we will prove some estimates which are needed for the proof of the upper bound, namely a variant of [GMS15, Lemma 2.8] which tells us that a word with few hamburger orders is very unlikely to contain too many flexible orders; and an estimate to the effect that the path Z^n is unlikely to stay close to the boundary of the first quadrant for a long time. In Section 5.3, we use the estimates of Section 4 and Section 5.2 to prove the upper bound in Theorem 1.10.

5.1. Lower bound.

In this subsection we will prove the lower bound in Theorem 1.10. The content of this subsection appeared in an earlier version of [GMS15], but was moved to its present location to present a more logical progression of ideas. We thank Cheng Mao for his contributions to this subsection.

First we need to estimate an appropriate probability for Brownian motion.

Lemma 5.1. *Fix a constant $C > 1$. Let $z \in [C^{-1}, C]^2$. Let Z be a Brownian motion as in* (8) *(started from 0). For $\delta > 0$, let F^z_δ be the event that $U(t) \geq -\delta^{1/2}$ and $V(t) \geq -\delta^{1/2}$ for each $t \in [0,1]$; and $|Z(1) - z| \leq \delta^{1/2}$. Then $\mathbb{P}\left(F^z_\delta\right) \succeq \delta^{\mu+1}$, where μ is as in* (11) *and the implicit constant depends on C, but not z or δ.*

Proof. Let $\widetilde{E}_\delta$ be the event that $U(t) \geq -\delta^{1/2}$ and $V(t) \geq -\delta^{1/2}$ for each $s \in [0, 1/2]$; and $Z(1/2) \in [C^{-1}, C]^2$. By [GMS15, Lemma 2.2] and scale invariance (see also [Shi85, Section 4]), we have $\mathbb{P}\left(\widetilde{E}_\delta\right) \asymp \delta^\mu$. The conditional law of $Z|_{[1/2,1]}$ given $Z|_{[0,1/2]}$ is that of a Brownian motion with covariances as in (8) started from $Z(1/2)$. On the event $\widetilde{E}_\delta$, the probability that such a Brownian motion stays in the first quadrant until time 1 and satisfies $|Z(1) - z| \leq \delta^{1/2}$ is proportional to δ. Hence $\mathbb{P}\left(F^z_\delta \,|\, \widetilde{E}_\delta\right) \asymp \delta$. The statement of the lemma follows. □

Proof of Theorem 1.10, lower bound. We find it more convenient to prove the lower bound with $X(-2n,-1)$ in place of $X(1,2n)$ (which we can do by translation invariance). Fix $\delta>0$ and $C>100$, to be chosen later. Let $\Bbbk_n$ be the smallest $k\in\mathbb{N}$ such that $\delta^k n\le 1$ and for $k\in[0,\Bbbk_n]_{\mathbb{Z}}$ let $m_n^k=\lfloor\delta^k n\rfloor$. Let μ be as in (11) and fix $\nu\in(1-\mu,1/2)$.

Let b_0^H (resp. b_0^C) be the number of hamburgers (resp. cheeseburgers) in $X(-2n,-n-1)$. For $k\in[0,\Bbbk_n]_{\mathbb{Z}}$ let b_k^H (resp. b_k^C) be the number of hamburgers (resp. cheeseburgers) in $X(-2n,-m_n^k-1)$.

Let $G_{n,0}$ be the event that $X(-2n,-n-1)$ contains no burgers, at least $7C^{-1}n^{1/2}$ hamburger orders, at least $7C^{-1}n^{1/2}$ cheeseburger orders, and at most $Cn^{1/2}$ total orders. For $k\in[1,\Bbbk_n]_{\mathbb{Z}}$, let $G_{n,k}$ be the event that the following is true.

(1) $\mathcal{N}_\theta\left(X(-m_n^{k-1},-m_n^k-1)\right)\le 0\vee(C^{-1}(\delta^k n)^{1/2}-(\delta n)^{\nu(k-1)}-1)$ for $\theta\in\{\text{H},\text{C}\}$.
(2) $\mathcal{N}_{\boxed{\text{F}}}\left(X(-m_n^{k-1},-m_n^k-1)\right)\le(\delta n)^{\nu(k-1)}$.
(3) $b_{k-1}^H-4C^{-1}(\delta^k n)^{1/2}-(\delta n)^{\nu(k-1)}\le\mathcal{N}_\theta\left(X(m_n^{k-1}+1,m_n^k)\right)\le b_{k-1}^H-3C^{-1}(\delta^k n)^{1/2}-(\delta n)^{\nu(k-1)}$ for $\theta\in\{\boxed{\text{H}},\boxed{\text{C}}\}$.

By [GMS15, Lemma 2.3], if C is chosen sufficiently large then we have $\mathbb{P}(G_{n,0})\ge n^{-\mu+o_n(1)}$. By inspection, if $k\in[1,\Bbbk_n]_{\mathbb{Z}}$ and m_n^k is sufficiently large then on $G_{n,k}$, the word $X(-2n,-m_n^k-1)$ contains no orders and

$$C^{-1}(\delta^k n)^{1/2}\le b_k^H\le 6C^{-1}(\delta^k n)^{1/2}\quad\text{and}\quad C^{-1}(\delta^k n)^{1/2}\le b_k^C\le 6C^{-1}(\delta^k n)^{1/2}. \tag{60}$$

By [She11, Theorem 2.5], [GMS15, Corollary 5.2], and Lemma 5.1, we can choose $m_*\in\mathbb{N}$, independent of n, in such a way that there is a deterministic constant $q\in(0,1)$, independent of n and δ, such that whenever $k\in[1,\Bbbk_n]_{\mathbb{Z}}$ with $m_n^k\ge m_*$, (60) holds on the event $G_{n,k}$ and

$$\mathbb{P}\left(G_{n,k}\,|\,X_{-2n}\dots X_{-m_n^k}\right)\mathbb{1}_{G_{k-1}}\ge q\delta^{\mu+1}\mathbb{1}_{G_{n,k-1}}.$$

Let k_* be the largest $k\in[1,\Bbbk_n]_{\mathbb{Z}}$ for which $m_n^k\ge m_*$. Then

$$\mathbb{P}\left(\bigcap_{k=0}^{k_*}G_{n,k}\right)\ge q^{\Bbbk_n}\delta^{\Bbbk_n(\mu+1)}n^{-\mu+o_n(1)}\ge n^{-2\mu-1+o_n(1)+o_\delta(1)} \tag{61}$$

with the $o_\delta(1)$ independent of n. It follows from (60) that on $\bigcap_{k=0}^{k_*}G_{n,k}$, the word $X(-2n,-m_n^k-1)$ contains no orders and fewer than $m_n^{k_*}$ burgers. Since m_* does not depend on n and $m_n^{k_*}\le\delta^{-1}m_*$, we have

$$\mathbb{P}\left(X(1,2n)=\emptyset\,|\,\bigcap_{k=0}^{k_*}G_{n,k}\right)\succeq 1, \tag{62}$$

with the implicit constant independent of n. By combining (61) and (62) and using that δ is arbitrary, we obtain the upper bound in Theorem 1.10. □

5.2. **Some miscellaneous estimates.** In this subsection we will prove some estimates which are needed for the proof of the upper bound in Theorem 1.10. First we prove a variant of [GMS15, Lemma 2.8] where we consider the number of hamburger orders in a word, rather than the length of the word.

Lemma 5.2. *Let μ' be as in (11). For $m\in\mathbb{N}$ and $\nu>\mu'$, we have*

$$\mathbb{P}\left(\exists\, n\in\mathbb{N}\text{ such that }\mathcal{N}_{\boxed{H}}(X(1,n))\ge m\text{ and }\mathcal{N}_{\boxed{F}}(X(1,n))\ge\mathcal{N}_{\boxed{H}}(X(1,n))^\nu\right)=o_m^\infty(m). \tag{63}$$

Furthermore, for each fixed $\eta>0$ and each $n\in\mathbb{N}$, we have

$$\mathbb{P}\left(\exists\, j\in[1,n]_{\mathbb{Z}}\text{ such that }\mathcal{N}_{\boxed{F}}(X(-j,-1))\ge\mathcal{N}_{\boxed{H}}(X(-j,-1))^\nu\vee n^{\eta\nu}\right)=o_n^\infty(n). \tag{64}$$

To prove Lemma 5.2, we first require the following further lemma.

Lemma 5.3. *Let μ' be as in (11). For $m \in \mathbb{N}$, let $\widetilde{I}_m^H$ be the mth smallest $i \in \mathbb{N}$ for which the word $X(1,i)$ contains no hamburgers. Then for each $\nu > \mu'$, we have*

$$\mathbb{P}\left(\exists\, k \geq m \text{ such that } \mathcal{N}_{\boxed{F}}\left(X(1,\widetilde{I}_k^H)\right) \geq k^{2\nu}\right) = o_m^\infty(m).$$

Proof. For $l \in \mathbb{N}$, let N_l be the lth smallest $m \in \mathbb{N}$ such that $X(1,\widetilde{I}_m^H)$ contains no cheeseburgers (equivalently no burgers). Let $M_m := \sup\{l \in \mathbb{N} : N_l \leq m\}$. By [GMS15, Lemma 2.7], for each sufficiently large $C > 1$ we have

$$\mathbb{P}\left(\widetilde{I}_{N_1}^H \geq C^2 m^2,\, \mathcal{N}_{\boxed{\mathrm{H}}}\left(X(1,C^2m^2)\right) > m\right) \geq m^{-2\mu' + o_n(1)}. \tag{65}$$

Note that a hamburger can only be added to the stack at one of the times $\widetilde{I}_k^H$ for $k \in \mathbb{N}$. Hence on the event (65), we have $N_1 \geq m$, so

$$\mathbb{P}\left(N_l \geq m\right) \geq m^{-2\mu' + o_n(1)}.$$

Observe that $\mathcal{N}_{\boxed{\mathrm{F}}}\left(X(1,\widetilde{I}_m^H)\right)$ can increase by at most 1 when m increases by 1, and can increase only if $m = N_l$ for some $l \in \mathbb{N}$. The statement of the lemma therefore follows from [GMS15, Lemma 2.10]. □

Proof of Lemma 5.2. Define the times $\widetilde{I}_m^H$ as in Lemma 5.3. For $m \in \mathbb{N}$, let E_m^H be the event that $\widetilde{I}_m^H = \widetilde{I}_{m-1}^H + 1$ and $X_{\widetilde{I}_m^H} = \boxed{\mathrm{H}}$. The events E_m^H are independent and by (4), we have $\mathbb{P}\left(E_m^H\right) = 1/4$ for each $m \in \mathbb{N}$. By Hoeffding's inequality,

$$\mathbb{P}\left(\mathcal{N}_{\boxed{\mathrm{H}}}\left(X(1,\widetilde{I}_m^H)\right) \leq m/8\right) = o_m^\infty(m). \tag{66}$$

By summing this estimate over $[m,\infty)_{\mathbb{Z}}$, we find that the probability that there exists even a single $k \geq m$ for which $\mathcal{N}_{\boxed{\mathrm{H}}}\left(X(1,\widetilde{I}_k^H)\right) \leq k/8$ is of order $o_m^\infty(m)$.

Suppose there exists $n \in \mathbb{N}$ such that $h_n := \mathcal{N}_{\boxed{\mathrm{H}}}(X(1,n)) \geq m$ and $\mathcal{N}_{\boxed{\mathrm{F}}}(X(1,n)) \geq h_n^{2\nu}$. For each such n, let m_n be the largest $k \in \mathbb{N}$ such that $\widetilde{I}_k^H \leq n$. We clearly have $m_n \geq h_n$, so by the above estimate it holds except on an event of probability $o_m^\infty(m)$ that $m_n \in [h_n, 8h_n]_{\mathbb{Z}}$ for each such n. Therefore, for each such n we have $m_n \geq m$ and $\mathcal{N}_{\boxed{\mathrm{F}}}\left(X(1,\widetilde{I}_{m_n}^H)\right) \geq (m_n/8)^\nu$. We now obtain (63) by means of Lemma 5.3.

To obtain (64), we use translation invariance and (63) with $m = \lfloor n^{\eta\nu}/8 \rfloor$ to find that for each $j \in [1,n]_{\mathbb{Z}}$, the probability that $\mathcal{N}_{\boxed{\mathrm{H}}}(X(-j,-1)) \geq n^{\eta\nu}/8$ and $\mathcal{N}_{\boxed{\mathrm{F}}}(X(-j,-1)) \geq \mathcal{N}_{\boxed{\mathrm{H}}}(X(-j,-1))^\nu$ is $o_n^\infty(n)$. By (66), the probability that $\mathcal{N}_{\boxed{\mathrm{F}}}(X(-j,-1)) \geq n^{\eta\nu}$ and $\mathcal{N}_{\boxed{\mathrm{H}}}(X(-j,-1)) < n^{\eta\nu}/8$ if also of order $o_n^\infty(n)$. We then sum over all $j \in [1,n]_{\mathbb{Z}}$. □

Next we will prove a result to the effect that it is very unlikely that the path $n \mapsto D(n)$ of Section 1.1 stays close to the coordinate axes for a long time.

Lemma 5.4. *Define $D = (d, d^*)$ as in Notation 1.6. There is a constant $a_0 > 0$, depending only on p, such that for each $n \in \mathbb{N}$ and $r > 0$, we have*

$$\mathbb{P}\left(\sup_{j \in [1,n]_{\mathbb{Z}}} |d\left(X(-j,-1)\right)| \wedge |d\left(X(-j,-1)\right)| \leq r\right) \leq e^{-a_0 r^{-2} n} + o_n^\infty(n). \tag{67}$$

Proof. To lighten notation, let

$$G_n(r) := \left\{\sup_{j \in [1,n]_{\mathbb{Z}}} |d\left(X(-j,-1)\right)| \wedge |d\left(X(-j,-1)\right)| \leq r\right\}.$$

Let μ be as in (11) and fix $\nu_1 < \nu_2 \in (1-\mu, 1/2)$. For $n \in \mathbb{N}$, let

$$F_n := \left\{ \sup_{j \in [n^{1-2\nu_2}, \infty)} j^{-\nu_1} \mathcal{N}_{\boxed{\mathrm{F}}}\left(X(-j,-1)\right) \leq 1 \right\}.$$

By [GMS15, Corollary 5.2], $\mathbb{P}(F_n) = 1 - o_n^\infty(n)$. Hence is suffices to bound $\mathbb{P}\left(G_n(r) \cap F_n\right)$.

For $n \in \mathbb{N}$, $r > 0$, and $k \in \left[1, r^{-2}n\right]_{\mathbb{Z}}$, let

$$E_n^k(r) := \left\{ \sup_{j \in [(k-1)r^2, kr^2]_{\mathbb{Z}}} \left(\left| d\left(X(-j, -(k-1)r-1)\right) \right| \wedge \left| d^*\left(X(-j,-(k-1)r-1)\right) \right| \right) \leq 3r+1 \right\}.$$

We claim that if $r \geq n^{\nu_1}$, then

$$G_n(r) \cap F_n \subset \bigcap_{k=\lfloor n^{1-2\nu_2} \rfloor + 1}^{\lfloor r^{-2}n \rfloor} E_n^k(r). \tag{68}$$

Indeed, suppose by way of contradiction that $G_n(r) \cap F_n$ occurs but there exists $k_* \in [n^{1-2\nu_2}+1, r^{-2}n]_{\mathbb{Z}}$ such that $E_n^{k_*}(r)$ fails to occur. Then there exists $j \in [(k_*-1)r^2, k_* r^2]_{\mathbb{Z}}$ for which $\left| d\left(X(-j, -(k_*-1)r^2+1)\right) \right| \geq 3r$ and $\left| d^*\left(X(-j, -(k_*-1)r^2+1)\right) \right| \geq 3r$. By definition of $G_n(r)$, either $\left| d\left(X(-(k_*-1)r^2, -1)\right) \right| \leq r$ or $\left| d^*\left(X(-(k_*-1)r^2, -1)\right) \right| \leq r$. By symmetry we can assume without loss of generality that we are in the former case. Then

$$\begin{aligned}
&\left| d\left(X(-j,-1)\right) \right| \\
&\quad \geq \left| d\left(X(-j, -(k_*-1)r^2-1)\right) \right| - \left| d\left(X(-(k_*-1)r^2, -1)\right) \right| - \mathcal{N}_{\boxed{\mathrm{F}}}\left(X(-(k_*-1)r^2,-1)\right) \\
&\quad \geq 2r + 1 - \mathcal{N}_{\boxed{\mathrm{F}}}\left(X(-(k-1)r^2, -1)\right).
\end{aligned}$$

On F_n, we have

$$\mathcal{N}_{\boxed{\mathrm{F}}}\left(X(-(k-1)r^2,-1)\right) \leq (k-1)^{\nu_1} r^{2\nu_1} \leq n^{\nu_1} \leq r.$$

Hence $\left| d\left(X(-j,-1)\right) \right| \geq r+1$, which contradicts occurrence of $G_n(r)$. Thus (68) holds.

The events $E_n^k(r)$ are independent. By [She11, Theorem 2.5], there is a $q > 0$, independent of n and r, such that $\mathbb{P}\left(E_n^k(r)^c\right) \geq q$ for each $n \in \mathbb{N}$, each $r \geq n^{\nu_1}$, and each $k \in [n^{1-2\nu_2}, r^{-2}n]_{\mathbb{Z}}$. Hence it follows from (68) that for $r \geq n^{\nu_1}$, we have

$$\mathbb{P}\left(G_n(r) \cap F_n\right) \leq (1-q)^{r^{-2}n - n^{1-2\nu_2} - 1} \leq e^{-a_0 r^{-2} n}$$

for an appropriate choice of a as in the statement of the lemma. It is clear that $\mathbb{P}\left(G_n(r) \cap F_n\right)$ is increasing in r, so for a general choice of $n \in \mathbb{N}$ and $r > 0$, we have

$$\mathbb{P}\left(G_n(r) \cap F_n\right) \leq e^{-a_0 r^{-2} n} \vee e^{-a_0 n^{1-2\nu_1}} \leq e^{-a_0 r^{-2} n} + o_n^\infty(n).$$

This yields (67). $\square$

5.3. Upper bound.

We are now ready to complete the proof of Theorem 1.10.

Proof of Theorem 1.10, upper bound. See Figure 2 for an illustration of the proof. We will prove the upper bound with $X(-2n,-1)$ in place of $X(1,2n)$. Let μ' be as in (11). Fix $\nu \in (\mu', 1/2)$, $\Bbbk \in \mathbb{N}$, and $\zeta \in (0, \nu^{\Bbbk})$, much smaller than $\nu^{\Bbbk}$. Let $\nu_\zeta \in (\mu', \nu)$ be chosen so that $\nu_\zeta(\nu^{\Bbbk} + 2\zeta) \leq \nu^{\Bbbk+1}$.

For $n \in \mathbb{N}$, let $N_n^0 := \lfloor n/2 \rfloor$ and let K_n^0 be minimum of $n+1$ and the smallest $j \geq N_n^0$ such $X(-j,-1)$ contains at least $n^{1/2-\zeta}$ hamburger orders and at least $n^{1/2-\zeta}$ cheeseburger orders. Inductively, if $k \in \mathbb{N}$ and N_n^{k-1} and K_n^{k-1} have been defined, define N_n^k and K_n^k as follows.

- Let N_n^k be the smallest $j \geq 2n - 2n^{(\nu^k+\zeta)}$ such that $X(-j,-1)$ contains at most $n^{\nu^k+\zeta}$ orders.
- Let K_n^k be the minimum of $\lfloor 2n - n^{2\nu^k} \rfloor$ and the smallest $j \geq N_n^k$ such that $X(-j,-1)$ contains at least $n^{\nu^k-\zeta}$ hamburger orders and at least $n^{\nu^k-\zeta}$ cheeseburger orders.

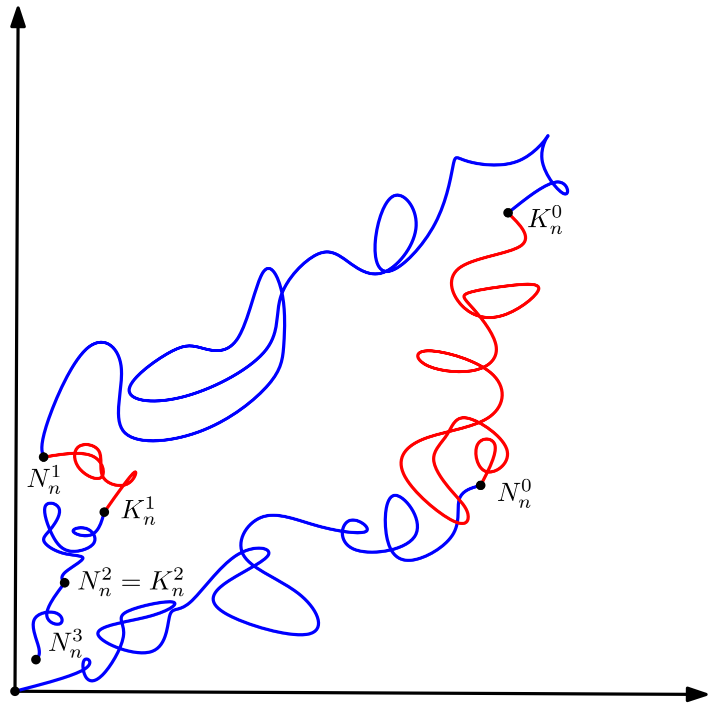

FIGURE 2. An illustration of the proof of the upper bound in Theorem 1.10. Fix $\nu \in (\mu', 1/2)$ and a small $\zeta > 0$. We first grow the word X backward up to a stopping time K_n^0 between $N_n^0 = \lfloor n/2 \rfloor$ and n such that it holds on event of probability $1 - o_n^\infty(n)$ that the word $X(-K_n^0, -1)$ contains at least $n^{1/2-\zeta}$ hamburger orders and at least $n^{1/2-\zeta}$ cheeseburger orders. Then we iterate the following procedure for $k \in \mathbb{N}$: grow until the first time N_n^k after K_n^{k-1} at which we have at most $n^{\nu^k+\zeta}$ orders; then, if we do not have at least $n^{\nu^k-\zeta}$ hamburger orders and at least $n^{\nu^k-\zeta}$ cheeseburger orders, grow until the first time K_n^k at which we have $n^{\nu^k-\zeta}$ hamburger orders and at least $n^{\nu^k-\zeta}$ cheeseburger orders. It holds except on an event of probability $o_n^\infty(n)$ that this latter time is finite by Lemma 5.4. Furthermore, using Lemma 5.2, we obtain that it holds except on an event of probability $o_n^\infty(n)$ that each of the words $X(-K_n^k, -1)$ contains at most $n^{\nu^{k+1}}$ flexible orders (this is our reason for considering increments of size n^{ν^k}). The figure shows three iterations of this procedure. Segments of the path corresponding to words of the form $X_{N_n^k} \dots X_{-K_n^{k-1}}$ are shown in blue and those corresponding to words of the form $X_{-K_n^k} \dots X_{-N_n^k-1}$ are shown in red. Note that it is possible that $K_n^k = N_n^k$ (which is the case for $k = 2$ here).

Observe that the times N_n^k and K_n^k are stopping times for the word X, read backward.

Let E_n^0 be the event that $K_n^0 \leq n$, $X(-K_n^0, -1)$ contains no burgers, and $\mathcal{N}_{\boxed{\mathrm{F}}}\left(X(-K_n^0, -1)\right) \leq n^\nu$.

For $k \in \mathbb{N}$, let E_n^k be the event that the following is true.

(1) $N_n^k \leq 2n - 2n^{2\nu^k}$ and $K_n^k < 2n - n^{2\nu^k}$.
(2) $X(-K_n^k, -1)$ contains at most $n^{\nu^k+2\zeta}$ orders.
(3) $X(-K_n^k, -1)$ contains no burgers.
(4) $\mathcal{N}_{\boxed{\mathrm{F}}}\left(X(-K_n^k, -1)\right) \leq n^{\nu^{k+1}}$.

We claim that for $k \in [0, \Bbbk]_{\mathbb{Z}}$, we have

$$\mathbb{P}\left((E_n^k)^c,\, X(-2n, -1) = \emptyset\right) \leq o_n^\infty(n). \tag{69}$$

The case where $k = 0$ follows from Lemma 5.4 together with [GMS15, Corollary 5.2]. Next suppose $k \in [1, \Bbbk]_{\mathbb{Z}}$. If $N_n^k > 2n - n^{2\nu^k}$ then $X(-N_n^k, -1)$ contains at least $n^{\nu^k + \zeta}$ orders. If also $X(-2n, -1) = \emptyset$, then $X(-2n, -N_n^k - 1)$ contains at least $n^{\nu^k + \zeta}$ burgers. Hence

$$\sup_{j \in [2n - 2n^{2\nu^k}, 2n]_{\mathbb{Z}}} |X(-j, -2n + 2n^{2\nu^k})| \geq n^{\nu^k + \zeta}.$$

By [She11, Lemma 3.13], the probability that this is the case is of order $o_n^\infty(n)$. By Lemma 5.2, the probability that $\mathcal{N}_{\boxed{\mathrm{H}}}\left(X(-N_n^k, -1)\right) \leq n^{\nu^k + \zeta}$ and $\mathcal{N}_{\boxed{\mathrm{F}}}\left(X(-K_n^k, -1)\right) \geq n^{\nu_\zeta(\nu^k + \zeta)}$ is of order $o_n^\infty(n)$. By our choices of ζ and $\Bbbk$ we have $\nu_\zeta(\nu^k + \zeta) \leq \nu^{k+1}$. Hence except on an event of probability $o_n^\infty(n)$, whenever $X(-2n, -1) = \emptyset$ we have

$$N_n^k \leq 2n - n^{2(\nu^k + \zeta)} \quad \text{and} \quad \mathcal{N}_{\boxed{\mathrm{F}}}\left(X(-K_n^k, -1)\right) \leq n^{\nu^{k+1}}. \tag{70}$$

Next we will show that it is very unlikely that $X(-2n, -1) = \emptyset$ and $K_n^k \geq 2n - n^{\nu^k}$. To this end, suppose (70) holds, $X(-2n, -1) = \emptyset$, and $K_n^k > N_n^k$. Then $X(-N_n^k, -1)$ either has fewer than $n^{\nu^k - \zeta}$ hamburger orders or fewer than $n^{\nu^k - \zeta}$ cheeseburger orders. Assume without loss of generality that we are in the former setting. Let $\widetilde{K}_n^k$ be the smallest $j \geq N_n^k + 1$ for which $X(-j, -N_n^k - 1)$ contains at least $n^{\nu^k - \zeta}$ hamburger orders. By (70) and Lemma 5.4, if $X(-2n, -1) = \emptyset$ then except on an event of probability $o_n^\infty(n)$ we have $\widetilde{K}_n^k \leq 2n - (3/2)n^{2\nu^k}$. If $K_n^k > \widetilde{K}_n^k$, then $X(-\widetilde{K}_n^k, -1)$ contains at most $n^{\nu^k - \zeta} + 1$ hamburger orders and at most $n^{\nu^k - \zeta}$ cheeseburger orders. By Lemma 5.2 (c.f. the argument above), if this is the case then except on an event of probability $o_n^\infty(n)$, the word $X(-\widetilde{K}_n^k, -1)$ contains at most $n^{\nu^{k+1}}$ flexible orders. Hence if $K_n^k \geq 2n - n^{\nu^k}$ and $X(-2n, -1) = \emptyset$, then except on an event of probability $o_n^\infty(n)$, we have

$$\sup_{j \in [\widetilde{K}_n^k + 1, 2n - n^{\nu^k}]_{\mathbb{Z}}} \left|d\left(X(-j, -N_n^k - 1)\right)\right| \wedge \left|d^*\left(X(-j, -N_n^k - 1)\right)\right| \leq n^{\nu^k - \zeta} + n^{\nu^{k+1}} + 1.$$

By Lemma 5.4, the probability that this is the case is of order $o_n^\infty(n)$. It follows that the probability that $X(-2n, -1) = \emptyset$ and condition 1 in the definition of E_n^k fails to occur is $o_n^\infty(n)$.

We have $K_n^k \geq N_n^k \geq 2n - n^{2(\nu^k + \zeta)}$ by definition, so [She11, Lemma 3.13] implies that except on an event of probability at most $o_n^\infty(n)$, we have

$$\sup_{j \in [K_n^k + 1, 2n]_{\mathbb{Z}}} |X(-j, -K_n^k - 1)| \leq n^{\nu^k + 2\zeta}. \tag{71}$$

If $X(-2n, -1) = \emptyset$, then $X(-2n, -K_n^k - 1)$ contains at least $|X(-K_n^k, -1)|$ burgers. From this and (71), we infer that condition 2 in the definition of E_n^k holds with probability at least $1 - o_n^\infty(n)$.

It is clear that condition 3 in the definition of E_n^k occurs whenever $X(-2n, -1) = \emptyset$. By Lemma 5.2, applied as in the argument leading to (70) (but with 2ζ in place of ζ) the probability that condition 2 in the definition of E_n^k occurs but condition 4 in the definition of E_n^k fails to occur is of order $o_n^\infty(n)$. This completes the proof of (69).

It remains to estimate $\mathbb{P}\left(\bigcap_{k=0}^{\Bbbk} E_n^k\right)$. Suppose $k \in [1, \Bbbk]_{\mathbb{Z}}$ and E_n^{k-1} occurs. By condition 2 in the definitions of E_n^{k-1} and E_n^k (and the definition of E_n^0 if $k = 1$), for each $k \in [1, \Bbbk]_{\mathbb{Z}}$, the word $X(-K_n^k, -K_n^{k-1} - 1)$ contains at least

$$\mathcal{N}_{\boxed{\mathrm{H}}}\left(X(-K_n^k, -1)\right) - n^{\nu^k + 2\zeta} \geq \begin{cases} (1 + o_n(1))n^{1/2 - \zeta} & k = 1 \\ (1 + o_n(1))n^{\nu^{k-1} - \zeta} & k \in [2, \Bbbk]_{\mathbb{Z}} \end{cases} \tag{72}$$

hamburgers, and similarly for cheeseburgers; and the total number of orders in $X(-K_n^k, -K_n^{k-1} - 1)$ is at most $n^{\nu^k + 2\zeta}$. Furthermore, by condition 4 in the definition of E_n^{k-1} (or the definition of E_n^0 if $k = 1$) and condition 3 in the definition of E_n^k, the word $X(-K_n^k, -1)$ contains at most

$$\mathcal{N}_{\boxed{\mathrm{H}}}\left(X(-K_n^k, -1)\right) + n^{\nu^k}$$

hamburgers, and similarly for cheeseburgers. Since $K_n^{k-1} \in \left[2n - n^{2(\nu^{k-1}+\zeta)}, 2n - n^{2\nu^{k-1}}\right]_{\mathbb{Z}}$, we infer from assertion 1 of Proposition 4.4 (applied with $h = \mathcal{N}_{\boxed{\text{H}}}\left(X(-K_n^k,-1)\right)+n^{\nu^k}$, $c = \mathcal{N}_{\boxed{\text{C}}}\left(X(-K_n^k,-1)\right)+n^{\nu^k}$, $2n-K_n^{k-1}$ in place of n, $k \asymp n^{2\nu^k+4\zeta}$, and $r \asymp n^{\nu^k+2\zeta}$) that for $k \in [2,\mathbb{k}]_{\mathbb{Z}}$,

$$\mathbb{P}\left(E_n^k \,|\, E_n^{k-1}\right) \leq \begin{cases} n^{-2(1+\mu)(1/2-\nu)+o_\zeta(1)+o_n(1)} & k=1 \\ n^{-2(1+\mu)(\nu^{k-1}-\nu^k)+o_\zeta(1)+o_n(1)} & k \in [2,\mathbb{k}]_{\mathbb{Z}} \end{cases}$$

with the $o_\zeta(1)$ depending on $\mathbb{k}$ but not n. By [GMS15, Proposition 5.1], we also have $\mathbb{P}\left(E_n^0\right) \leq n^{-\mu+o_n(1)}$. Therefore,

$$\mathbb{P}\left(\bigcap_{k=0}^{\mathbb{k}} E_n^k\right) \leq n^{-1-2\mu+2(1+\mu)\nu^{\mathbb{k}}+o_\zeta(1)+o_n(1)}.$$

By combining this with (69), we get

$$\mathbb{P}\left(X(-2n,-1)=\emptyset\right) \leq n^{-1-2\mu+2(1+\mu)\nu^{\mathbb{k}}+o_\zeta(1)+o_n(1)}.$$

By sending $\zeta \to 0$ and then $\mathbb{k} \to \infty$, we conclude. □

Appendix A. Proof of Proposition 3.7

In this appendix, we will prove Proposition 3.7, which is the key input in the proofs of Propositions 3.8 and 3.9. Throughout this appendix we continue to use the notation of Section 3.1.

A.1. **Brownian motion estimates.** In this section we will prove some estimates for Brownian motion which are related to the scaling limits of the quantities considered in Section 3.1. We start with some basic calculations for an unconditioned Brownian motion.

Lemma A.1. *Let B be a standard linear Brownian motion. For $u \in \mathbb{R}$, let τ_u be the last time $t \in [0,1]$ such that $B_s > u$ for each $s \in (t,1]$; or $\tau_u = 0$ if no such time exists. The density of τ_u restricted to the event $\tau_u > 0$ is given by*

$$\mathbb{P}\left(t < \tau_u < t+dt\right) = \frac{e^{-u^2/2t}}{\pi\sqrt{t(1-t)}}\,dt.$$

Proof. Let σ_u be the first time B hits u and set $\widetilde{B}_s := B_{s+\sigma_u} - B_{\sigma_u}$ and $\widetilde{\tau}_u := \sup\{s \in [0,1-\sigma_u] : \widetilde{B}_s = 0\}$. Then $\widetilde{B}$ is a standard linear Brownian motion independent from σ_u. The time $\widetilde{\tau}_u$ is equal to $\tau_u - \sigma_u$ on the event $\{\widetilde{B}_s > 0\,\forall s \in (\widetilde{\tau}_u, 1-\sigma_u]\}$, which by symmetry has probability $1/2$, even if we condition on $\widetilde{\tau}_u$. The conditional law of $\widetilde{\tau}_u$ given σ_u is given by the arcsine distribution,

$$\mathbb{P}\left(\widetilde{\tau}_u \leq t \,|\, \sigma_u\right) = \frac{2}{\pi}\arcsin\left(\sqrt{\frac{t}{1-\sigma_u}}\right).$$

The law of σ_u is given by

$$\frac{|u|}{\sqrt{2\pi}} s^{-3/2} e^{-u^2/2s}\,ds,$$

so we have

$$\begin{aligned}\mathbb{P}\left(0 < \tau_u \leq t\right) &= \frac{1}{2}\mathbb{E}\left(\mathbb{P}\left(0 < \widetilde{\tau}_u \leq t-\sigma_u\right) \,|\, \sigma_u\right) \\ &= \frac{|u|}{\sqrt{2}\pi^{3/2}} \int_0^t \arcsin\left(\sqrt{\frac{t-s}{1-s}}\right) s^{-3/2} e^{-u^2/2s}\,ds.\end{aligned}$$

By differentiating, we get that the density of τ_u is given by

$$\frac{|u|}{2\sqrt{2}\pi^{3/2}} \int_0^t \frac{s^{-3/2}e^{-u^2/2s}}{\sqrt{(1-t)(t-s)}}\,ds = \frac{e^{-u^2/2t}}{\pi\sqrt{t(1-t)}}.$$

□

Lemma A.2. *Let $Z = (U, V)$ be a correlated two-dimensional Brownian motion as in* (8)*. Fix $T > 0$. For $u \in \mathbb{R}$, let $\widetilde{\tau}_u$ be the last time $t \in [0, T]$ such that $U(s) > u$ for each $s \in (t, T]$; or $\widetilde{\tau}_u = 0$ if no such time exists. The joint density of $\widetilde{\tau}_u$ and $V(\widetilde{\tau}_u)$ on the event $\{\widetilde{\tau}_u > 0\}$ is given by*

$$\mathbb{P}\left(t < \widetilde{\tau}_u < t + dt,\, v < V(\widetilde{\tau}_u) < v + dv\right) = \frac{a_0}{t\sqrt{T-t}} \exp\left(-\frac{a_2 u^2 - a_3(v - a_1 u)^2}{t}\right) dt\, dv \tag{73}$$

for constants $a_0, a_1, a_2, a_3 > 0$ depending only on p.

Proof. First consider the case where $T = 1$. The density of $\widetilde{\tau}_u$ with respect to Lebesgue measure on the event $\{\widetilde{\tau}_u > 0\}$ is computed in Lemma A.1. The Brownian motion V can be written as the sum of a constant a_1 (depending only on p) times U; and a Brownian motion $\widetilde{V}$ which is independent from U. On the event $\{\widetilde{\tau}_u > 0\}$, we therefore have $V(\widetilde{\tau}_u) = \widetilde{V}(\widetilde{\tau}_u) + a_1 u$. Hence the conditional law of $V(\widetilde{\tau}_u)$ given $\widetilde{\tau}_u$ on the event $\{\widetilde{\tau}_u > 0\}$ is that of a Gaussian random variable with mean $a_1 u$ and variance a constant depending only on p times $\widetilde{\tau}_u$. This yields the formula (73) in the case $T = 1$. For general $T > 0$, we have by Brownian scaling that

$$(\widetilde{\tau}_u, V(\widetilde{\tau}_u)) \overset{d}{=} \left(T\widetilde{\tau}_{T^{-1/2}u}(1), T^{1/2} V(\widetilde{\tau}_{T^{-1/2}u}(1))\right)$$

where $\tau_u(1)$ denotes the time τ_u defined with $T = 1$. This yields the formula (73) in general. □

From Lemma A.2, we obtain an estimate for a Brownian motion conditioned to stay in the first quadrant.

Lemma A.3. *Let $\widehat{Z} = (\widehat{U}, \widehat{V})$ have the law of a correlated Brownian motion as in* (8) *conditioned to stay in the first quadrant until time 1. For $u > 0$, let τ_u be the last time $t \in [0, 1]$ such that $\widehat{U}(s) > u$ for each $s \in (t, 1]$; or $\tau_u = 0$ if no such time exists. For each $(u, v) \in (0, \infty)^2$, $\epsilon_1, \epsilon_2 > 0$, and $a \in [0, 1)$, we have*

$$\mathbb{P}\left((\tau_u, \widehat{V}(\tau_u)) \in [1 - \epsilon_1^2, 1 - a\epsilon_1^2] \times [v - \epsilon_2, v + \epsilon_2]\right) \asymp \epsilon_1 \epsilon_2 \tag{74}$$

with the implicit constant in the upper bound independent of ϵ_1, ϵ_2, and a and uniform for $(u, v) \in (0, \infty)^2$; and the implicit constant in the lower bound independent of ϵ_1 and ϵ_2 and uniform for $(u, v) \in [C^{-1}, C]$ for each $C > 1$.

Furthermore, for $\zeta > 0$ and $b > 0$, let $F_{u,b}(\zeta)$ to be the event that there is a $t \in [0, \tau_u]$ such that $\widehat{U}(t) \le \zeta$ and $\widehat{V}(t) \ge b$. Then we have

$$\mathbb{P}\left((\tau_u, \widehat{V}(\tau_u)) \in [1 - \epsilon_1^2, 1] \times [v - \epsilon_2, v + \epsilon_2],\, F_{u,b}(\zeta)\right) \le \epsilon_1 \epsilon_2 o_\zeta(1) \tag{75}$$

with the $o_\zeta(1)$ depending only on ζ and b and the implicit constant uniform for $(u, v) \in (0, \infty)^2$.

Proof. We first prove a slightly more general statement with $\widehat{Z}$ replaced by the unconditioned correlated Brownian motion $Z = (U, V)$. For $u \in \mathbb{R}$ and $T > 0$, let $\widetilde{\tau}_u(T)$ be the last time $t \in [0, T]$ such that $\widehat{U}(s) > u$ for each $s \in (t, T]$; or $\widetilde{\tau}_u(T) = 0$ if no such time exists. For $\epsilon_1, \epsilon_2 > 0$, $u, v > 0$, $a \in [0, 1)$, and $T > 0$, let

$$E_{u,v}^{\epsilon_1,\epsilon_2}(T) := \left\{(\widetilde{\tau}_u(T), V(\widetilde{\tau}_u(T))) \in [T - \epsilon_1^2, T - a\epsilon_1^2] \times [v - \epsilon_2, v + \epsilon_2]\right\}.$$

It follows from Lemma A.2 that

$$\mathbb{P}\left(E_{u,v}^{\epsilon_1,\epsilon_2}(T)\right) \asymp \epsilon_1 \epsilon_2 \tag{76}$$

with the implicit constant in the upper bound depending only on p (and in particular uniform for $(u, v) \in \mathbb{R}^2$) and the implicit constant in the lower bound depending only on a and uniform for (u, v) in compact subsets of $(0, \infty)^2$.

It remains to transfer the estimate to $\widehat{Z}$. For $\epsilon_1, \epsilon_2 > 0$ and $u, v > 0$, let

$$\widehat{E}_{u,v}^{\epsilon_1,\epsilon_2} := \left\{(\tau_u, \widehat{V}(\tau_u)) \in [1 - \epsilon_1^2, 1 - a\epsilon_1^2] \times [v - \epsilon_2, v + \epsilon_2]\right\}.$$

Let σ be a stopping time for $\widehat{Z}$. For $z \in (0,\infty)^2$, let $\mathbb{P}^z$ denote the law of the correlated Brownian motion (8) started from z and for $T > 0$ let G_T be the event that this Brownian motion stays in the first quadrant until time $1 - t_0$.

By the Markov property of $\widehat{Z}$ and (74) for the unconditioned Brownian motion Z, we have

$$\mathbb{P}\left(E_{u,v}^{\epsilon_1,\epsilon_2},\, \tau_u > \sigma \,|\, \widehat{Z}|_{[0,\sigma]}\right) = \mathbb{P}^{\widehat{Z}(t_0)}\left(E_{u,v}^{\epsilon_1,\epsilon_2}(1-\sigma) \,|\, G_{1-\sigma}\right) \tag{77}$$

Thus

$$\mathbb{P}\left(E_{u,v}^{\epsilon_1,\epsilon_2},\, \tau_u > \sigma \,|\, \widehat{Z}|_{[0,\sigma]}\right) \leq \frac{\mathbb{P}^{\widehat{Z}(t_0)}\left(E_{u,v}^{\epsilon_1,\epsilon_2}(1-t_0) \,|\, \widehat{Z}|_{[0,\sigma]}\right)}{\mathbb{P}^{\widehat{Z}(t_0)}\left(G_{1-\sigma} \,|\, \widehat{Z}|_{[0,\sigma]}\right)}$$
$$= \frac{\mathbb{P}^{0}\left(E_{u-\widehat{U}(\sigma),v-\widehat{V}(\sigma)}^{\epsilon_1,\epsilon_2}(1-\sigma) \,|\, \widehat{Z}|_{[0,\sigma]}\right)}{\mathbb{P}^{\widehat{Z}(t_0)}\left(G_{1-\sigma} \,|\, \widehat{Z}|_{[0,\sigma]}\right)}.$$

By (76) and [Shi85, Equation 4.2], this quantity is at most a constant (depending only on p) times

$$\epsilon_1\epsilon_2|\widehat{Z}(\sigma)|^{-2\mu}(1-\sigma)^{\mu} \leq \epsilon_1\epsilon_2|\widehat{Z}(\sigma)|^{-2\mu}.$$

with μ as in (11). If we take $\sigma = 1/2$, say, then by [Shi85, Equation 4.2], $\mathbb{E}\left(|\widehat{Z}(\sigma)|^{-2\mu}\right) < \infty$. We thus obtain the upper bound in (74).

On the other hand, if we let σ be the smallest $t \in [0,1]$ for which $\widehat{U}(t) \leq \zeta$ and $\widehat{V}(t) \geq b$ (or $\sigma = \infty$ if no such u exists) then with $F_{u,b}(\zeta)$ as in (75), we have

$$F_{u,b}(\zeta) \subset \{\sigma < \infty\}.$$

Since $\widehat{Z}$ a.s. does not hit the boundary of the first quadrant after time 0, it follows that $\mathbb{P}(\sigma < \infty) = o_\zeta(1)$, at a rate depending only on b. By choosing this σ in our calculation above, we find that the event of (75) has probability at most a constant times

$$\epsilon_1\epsilon_2\mathbb{E}\left(|\widehat{Z}(\sigma)|^{-2\mu}\mathbb{1}_{(\sigma<\infty)}\right) \leq \epsilon_1\epsilon_2 b^{2\mu}\mathbb{P}(\sigma<\infty) = \epsilon_1\epsilon_2 o_\zeta(1).$$

It remains to prove the lower bound in (74). To this end, fix $C > 1$. Observe that for each $T > 0$, the regular conditional law of $Z|_{[0,\widetilde{\tau}_u(T)]}$ given an realization of $(\widetilde{\tau}_u(T), Z(\widetilde{\tau}_u(T)))$ for which $\widetilde{\tau}_u(T) > 0$ is that of a Brownian bridge from 0 to $\widetilde{Z}(\widetilde{\tau}(T))$ in time $\widetilde{\tau}_u(T)$. Hence if we let A be the event that $\inf_{t\in[0,\widetilde{\tau}_u(T)]} U(t)$ and $\inf_{t\in[0,\widetilde{\tau}_u(T)]} V(t)$ are each at least $-(4C)^{-1}$, then for $(u,v) \in [(2C)^{-1}, 2C]^2$, $\mathbb{P}\left(A \,|\, E_{u,v}^{\epsilon_1,\epsilon_2}(T)\right)$ is at least a positive constant depending only on C and T. By (76), for such a pair (u,v) we have

$$\mathbb{P}\left(E_{u,v}^{\epsilon_1,\epsilon_2}(T) \cap A\right) \succeq \epsilon_1\epsilon_2$$

with implicit constants depending only on C and T. By [Shi85, Theorem 2], for $t_0 \in (0,1)$ we have that $\mathbb{P}\left(\widehat{Z}(t_0) \in [(4C)^{-1}, (2C)^{-1}]^2\right)$ is at least a positive constant depending only on t_0 and C. By (77),

$$\begin{aligned}\mathbb{P}\left(E_{u,v}^{\epsilon_1,\epsilon_2}\right) &\succeq \mathbb{E}\left(\mathbb{P}^{\widehat{Z}(t_0)}\left(E_{u,v}^{\epsilon_1,\epsilon_2}(1-t_0) \,|\, G_{1-t_0}\right)\mathbb{1}_{(\widehat{Z}(t_0)\in[(4C)^{-1},(2C)^{-1}]^2)}\right) \\ &\succeq \mathbb{E}\left(\mathbb{P}^{0}\left(E_{u,v}^{\epsilon_1-\widehat{U}(t_0),\epsilon_2-\widehat{U}(t_0)}(1-t_0)\cap A\right)\mathbb{1}_{(\widehat{Z}(t_0)\in[(4C)^{-1},(2C)^{-1}]^2)}\right) \\ &\succeq \epsilon_1\epsilon_2.\end{aligned}$$

□

We want to use the above lemmas to prove estimates for the word X conditioned on $\{I > n\}$. To do this we need the following fact.

Lemma A.4. *Let $\widehat{Z} = (\widehat{U}, \widehat{V})$ have the law of a correlated Brownian motion as in (8) conditioned to stay in the first quadrant until time 1. For $u > 0$, let τ_u be the last time $t \in [0,1]$ such that $\widehat{U}(s) > u$ for each $s \in (t,1]$; or $\tau_u = 0$ if no such time exists. Define the times $K_{n,m}^H$ and the quantities $Q_{n,m}^H$ as in Section 3.1. For each $\epsilon > 0$ and $C > 1$, there exists $n_* = n_*(\epsilon, C) \in \mathbb{N}$ such that for* *each $n \geq n_*$ and each $m \in [C^{-1}n^{1/2}, Cn^{1/2}]_{\mathbb{Z}}$, the Prokhorov distance between the conditional law of $(Z^n, n^{-1}K^H_{n,m}, n^{-1/2}Q^H_{n,m})$ given $\{I > n\}$ and the law of $(\widehat{Z}, \tau_{m/n^{1/2}}, \widehat{V}(\tau_{m/n^{1/2}}))$ is at most ϵ.*

Proof. By [GMS15, Theorem A.1] and the Skorokhod theorem, we can find a coupling of a sequence of words $(\widehat{X}^n)$ distributed according to the conditional law of the word X given $\{I > n\}$; and the path $\widehat{Z}$ such that in this coupling $\widehat{Z}^n \to \widehat{Z}$ uniformly a.s. on $[0,1]$, where $\widehat{Z}^n = (\widehat{U}^n, \widehat{V}^n)$ is the path constructed from the word $\widehat{X}^n$ as in (7). For $u > 0$, let τ^n_u be the last time $t \in [0,1]$ for which $U^n(s) > u$ for each $s \in (t,1]$; and otherwise let $\tau^n_u = 0$. If we construct the pair $(n^{-1}K^H_{n,m}, n^{-1/2}Q^H_{n,m})$ as in Section 3.1 from the word $\underline{X}^n$, then for $m \in \mathbb{N}$,

$$(n^{-1}K^H_{n,m}, n^{-1/2}Q^H_{n,m}) = (\tau^n_{m/n^{1/2}} + o_n(1), \widehat{V}^n(\tau^n_{m/n^{1/2}}) + o_n(1)).$$

Hence it suffices to show that in our coupling, $(\tau^n_u, \widehat{V}^n(\tau^n_u)) \to (\tau_u, \widehat{V}(\tau_u))$ in probability, uniformly for $u \in [C^{-1}, C]$.

Fix $\epsilon > 0$ and $C > 1$. We observe the following.

(1) By equicontinuity, we can find $\delta_1 \in (0, \epsilon]$ such that with probability at least $1 - \epsilon$, we have $|\widehat{Z}(t) - \widehat{Z}(s)| \leq \epsilon$ and $|\widehat{Z}^n(t) - \widehat{Z}^n(s)| \leq \epsilon$ whenever $n \in \mathbb{N}$ and $s, t \in [0,1]$ with $|s - t| \leq \delta_1$.
(2) The conditional law of $\widehat{U}(\cdot + \tau_u) - u$ given τ_u on the event $\{\tau_u > 0\}$ is that of a one-dimensional Brownian motion conditioned to stay positive until time 1. Hence we can find $\delta_2 \in (0, \delta_1/2]$, independent from u, such that with probability at least $(1-\epsilon)\mathbb{P}(\tau_u 0)$, we have $\tau_u > 0$ and $\widehat{U}(t) \geq u + \delta_2$ for each $t \in [\tau_u + \delta_1, 1]$.
(3) We can find $\delta_3 > 0$, independent from $u \in [C^{-1}, C]$, such that for $u \in [C^{-1}, C]$ it holds with probability at least $(1-\epsilon)\mathbb{P}(\tau_u > 0)$ that $\tau_u > 0$ and $\inf_{t \in [\tau_u - \delta_1, \tau_u]} \widehat{U}(t) \leq u - \delta_3$.
(4) By uniform convergence, we can find $N_1 \in \mathbb{N}$ such that with probability at least $1 - \epsilon$ it holds for $n \geq N_1$ that $|\widehat{Z}(t) - \widehat{Z}^n(t)| \leq (\delta_2 \wedge \delta_3)/2$ for each $t \in [0,1]$.

Let E_u be the event that $\tau_u > 0$ and the above four conditions hold, so that $\mathbb{P}(E_u) \geq (1-2\epsilon)\mathbb{P}(\tau_u > 0) - 2\epsilon$. By conditions 2 and 4, on E_u, we have $\widehat{Z}^n(t) \geq u + \delta_2/2$ for each $n \geq N_1$ and $t \geq \tau_u + \delta_1$. In particular, $\widehat{Z}^n(\tau_u + \delta_1) \geq u + \delta_2/2$. By combining this with condition 3 and using continuity of $\widehat{Z}^n$, we find that there exists $t \in [\tau_u - \delta_1, \tau_u + \delta_1]$ such that $\widehat{Z}^n(t) = u$. Since $\widehat{Z}^n(t) > u$ for $t \geq \tau_u + \delta_1$, we obtain $\tau^n_u \in [\tau_u - \delta_1, \tau_u + \delta_1]$. By conditions 1 and 4, on E_u we have $|\widehat{V}^n(\tau^n_u) - \widehat{V}(\tau_u)| \leq 2\epsilon$. Hence for $u > 0$ and $n \geq N_1$,

$$\mathbb{P}\left(|(\tau^n_u, \widehat{V}^n(\tau^n_u)) - (\tau_u, \widehat{V}^n(\tau_u))| \leq 3\epsilon,\, \tau_u > 0\right) \geq (1-2\epsilon)\mathbb{P}(\tau_u > 0) - 2\epsilon. \tag{78}$$

We have

$$\{\tau_u = 0\} = \left\{ \sup_{t \in [0,1]} \widehat{U}(t) < u \right\}.$$

Since $\mathbb{P}\left(\sup_{t \in [0,1]} \widehat{U}(t) \in [u - \delta_4, u]\right) \to 0$ as $\epsilon \to 0$, uniformly in u, it follows that we can find a deterministic $N_2 \in \mathbb{N}$ such that for $n \geq N_2$,

$$\mathbb{P}\left(\{\tau^n_u = 0,\, \tau_u > 0\} \cup \{\tau^n_u > 0,\, \tau_u = 0\}\right) \leq \epsilon.$$

It now follows from (78) that for $n \geq N_1 \vee N_2$,

$$\mathbb{P}\left(|(\tau^n_u, \widehat{V}^n(\tau^n_u)) - (\tau_u, \widehat{V}^n(\tau_u))| \leq 3\epsilon\right) \geq 1 - 5\epsilon.$$

Since ϵ is arbitrary, we conclude. □

A.2. **Proof of the proposition.** In this subsection, we will establish Proposition 3.7.

We continue to use the notation introduced in Section 3.1 plus the notation in the statement of Proposition 3.7 (recall in particular the notation $m^\delta_h := \lfloor (1-\delta)h \rfloor$). We also introduce the following additional notation. For $\delta \in (0, 1/2)$ and $C > 1$, let $\Bbbk_\delta = \Bbbk_\delta(C)$ be the smallest $k \in \mathbb{N}$ for which

$2^k\delta \geq 2C^2$. For $n\in\mathbb N$, $(k_1,k_2)\in(-\infty,\Bbbk_\delta]_{\mathbb Z}\times[1,\Bbbk_\delta)_{\mathbb Z}$, and $(h,c)\in\mathbb N^2$, let $\mathcal U_n^{k_1,k_2}(\delta,h,c)$ be the set of those pairs $(r,l)\in\mathbb N^2$ such that

$$n-r\in\left[2^{2(k_1-1)}\delta^2h^2, 2^{2k_1}\delta^2h^2\right]_{\mathbb Z}\quad\text{and}\quad |l-c|\in\left[2^{k_2-1}\delta h, 2^{k_2}\delta h\right]_{\mathbb Z}.$$

Also let $\mathcal U_n^{k_1,\Bbbk_\delta}(\delta,h,c)$ be the set of those pairs $(r,l)\in\mathbb N^2$ such that

$$n-r\in\left[2^{2(\Bbbk_\delta-1)}\delta^2h^2, 2^{2k_1}\delta^2h^2\right]_{\mathbb Z}\quad\text{and}\quad |l-c|\in\left[2^{\Bbbk_\delta}\delta h, \infty\right)_{\mathbb Z};$$

and let $\mathcal U_n^{k_1,0}(\delta,h,c)$ be the set of those pairs $(r,l)\in\mathbb N^2$ such that

$$n-r\in\left[2^{2(\Bbbk_\delta-1)}\delta^2h^2, 2^{2k_1}\delta^2h^2\right]_{\mathbb Z}\quad\text{and}\quad |l-c|\in\left[0, \frac12\delta^2h^2\right]_{\mathbb Z}.$$

Lemma A.5. *Fix $C>1$. For $\delta\in(0,1/2)$, $n\in\mathbb N$, $(h,c)\in\left[C^{-1}n, Cn\right]_{\mathbb Z}$, and $k_1,k_2\in[1,\Bbbk_\delta]_{\mathbb Z}$, we have*

$$\mathbb P\left((K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta})\in\mathcal U_n^{k_1,k_2}(\delta,h,c)\,|\, I>n\right)\asymp 2^{k_1+k_2}\delta^2+o_n(1) \tag{79}$$

with the implicit constant depending only on C and the $o_n(1)$ depending only on n, δ, and C. Furthermore, for $\zeta>0$ and $b>0$, let $F^H_{n,b}(\zeta)$ be the event that there is an $i\in[1,K^H_{n,m_h^\delta}]_{\mathbb Z}$ such that $\mathcal N_{C}(X(1,i))\geq bn^{1/2}$ and $\mathcal N_{H}(X(1,i))\leq\zeta n^{1/2}$. Then we have

$$\mathbb P\left((K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta})\in\mathcal U_n^{k_1,k_2}(\delta,h,c),\, F^H_{n,m_c^\delta}(\zeta)\,|\, I>n\right)= 2^{k_1+k_2}\delta^2 o_\zeta(1)+o_n(1), \tag{80}$$

with the $o_\zeta(1)$ depending only on ζ and C and the $o_n(1)$ depending only on n, ζ, δ, and C.

Proof. First we prove (79). Fix $\epsilon>0$. Define $\widehat Z=(\widehat U,\widehat V)$ and τ_u for $u>0$ as in Lemma A.3. That lemma implies that for each h,c,δ,k_1,k_2 as in the statement of the lemma, we have

$$\mathbb P\left((\tau_{m_h^\delta/n^{1/2}}, \widehat V(\tau_{m_h^\delta/n^{1/2}}))\in[1-2^{2k_1}\delta^2, 1-2^{-2k_1}\delta^2]\times[n^{-1/2}c-2^{k_2}\delta, n^{-1/2}c+2^{k_2}\delta]\right)\asymp 2^{k_1+k_2}\delta^2, \tag{81}$$

with the implicit constant depending only on C. By Lemma A.2 together with the same argument used in the proof of Lemma A.3, we can find $\alpha>0$, depending on δ and ϵ but not on $h,c,$, k_1, or k_2, such that for each $(h,c)\in\left[C^{-1}n,Cn\right]_{\mathbb Z}$ and $k_1,k_2\in[1,\Bbbk_\delta]_{\mathbb Z}$, we have

$$\mathbb P\left((\tau_{m_h^\delta/n^{1/2}}, \widehat V(\tau_{m_h^\delta/n^{1/2}}))\in B_\alpha\left(\partial\left([1-2^{2k_1}\delta^2, 1-2^{-2k_1}\delta^2]\times[n^{-1/2}c-2^{k_2}\delta, n^{-1/2}c+2^{k_2}\delta]\right)\right)\right)\leq\epsilon.$$

By Lemma A.4, we can find $n_*=n_*(\epsilon,\alpha,\delta,C)\in\mathbb N$ such that for each $n\geq n_*$ and each $m\in[C^{-1}n^{1/2}, Cn^{1/2}]_{\mathbb Z}$, the Prokhorov distance between the conditional law of $(Z^n, n^{-1}K^H_{n,m}, n^{-1/2}Q^H_{n,m})$ given $\{I>n\}$ and the law of $(\widehat Z, \tau_{m/n^{1/2}}, \widehat V(\tau_{m/n^{1/2}}))$ is at most $\delta^{100}(\epsilon\wedge\alpha)$ (say). It follows that for $n\geq n_*$,

$$2^{k_1+k_2}\delta^2-2\epsilon\preceq\mathbb P\left((K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta})\in\mathcal U_n^{k_1,k_2}(\delta,h,c)\,|\, I>n\right)\preceq 2^{k_1+k_2}\delta^2+2\epsilon,$$

with implicit constants as in (81). This proves (79). We similarly obtain (80) using (75) of Lemma A.3. $\square$

Lemma A.6. *For each $C>1$, there exists $\delta_*>0$ (depending only on C) such that for each $\delta\in(0,\delta_*]$, there exists $n_*=n_*(\delta,C)\in\mathbb N$ such that for $n\geq n_*$ and $(h,c)\in\left[C^{-1}n^{1/2}, Cn^{1/2}\right]^2_{\mathbb Z}$, we have*

$$\mathbb P\left(\mathcal E_n^{h,c},\, (K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta})\in\mathcal U_n^{1,1}(\delta,h,c)\,|\, I>n\right)\succeq n^{-1}$$

with the implicit constant depending only on C.

Proof. Set $r_h^\delta := h - m_h^\delta$. Let $(h,c) \in \left[C^{-1}n^{1/2}, Cn^{1/2}\right]_{\mathbb{Z}}^2$. Let x be a realization of $X_1 \dots X_{K^H_{n,m_h^\delta}}$ for which $0 < K^H_{n,m_h^\delta} < I$ and $(K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta}) \in \mathcal{U}_n^{1,1}(\delta,h,c)$. Let $\rfloor(x) = \mathcal{N}_{C}(\mathcal{R}(x))$ be the corresponding realization of Q^H_{n,m_h^δ}. Define $(J^H_{n,r_h^\delta}, L^H_{n,r_h^\delta})$ as in Section 3.1.

Since x is chosen so that $(K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta}) \in \mathcal{U}_n^{1,1}(\delta,h,c)$, we have $n - |x| \asymp \delta^2 n$ (with the implicit constant depending only on C); and if $\delta \in (0,(4C)^{-1})$, then $\mathfrak{c}(x) \geq (2C)^{-1}n^{-1/2}$. By Lemmas 2.2 and 2.10, it follows that we can find $\delta_* \in (0, \widetilde{\delta}_*]$, depending only on C, such that for $\delta \in (0,\delta_*]$, we have

$$\mathbb{P}\left((J^H_{n,r_h^\delta}, L^H_{n,r_h^\delta}) = (n - |x| - 1, c - \mathfrak{c}(x)), \sup_{j \in [1, n-|x|-1]} |X(n-j,n)| \leq \mathfrak{c}(x) \right) \succeq \delta^{-3}h^{-3} \succeq \delta^{-3}n^{-3/2}$$

with the implicit constant depending only on C.

By (31) and Lemma 2.6, we therefore have

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{E}_n^{h,c} \,|\, X_1 \dots X_{K^H_{n,m_h^\delta}} = x,\, I > n \right) \\
&\quad \succeq (n-|x|)^{1/2} \mathbb{P}\left((J^H_{n,r_h^\delta}, L^H_{n,r_h^\delta}) = (n-|x|-1, c - \mathfrak{c}(x)),\, \mathcal{N}_{\boxed{C}}\left(X(n - J^H_{n,h-m}, n)\right) \leq \mathfrak{c}(x) \right) \\
&\quad \succeq \delta^{-2}n^{-1}. \qquad (82)
\end{aligned}$$

with the implicit constant depending only on C. By Lemma A.5, for any $\delta \in (0,\delta_*]$, we can find $n_* = n_*(\delta, C) \in \mathbb{N}$ such that for $n \geq n_*$ and (h,c) as in the statement of the lemma, we have

$$\mathbb{P}\left((K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta}) \in \mathcal{U}_n^{1,1}(\delta,h,c) \,|\, I > n \right) \succeq \delta^2$$

with the implicit constant depending only on C. By combining this with (82), we obtain the statement of the lemma. □

Lemma A.7. *Fix $C > 1$. For $(h,c) \in \mathbb{N}^2$, $k_1, k_2 \in \mathbb{N}$, and $\delta > 0$, define $\Bbbk_\delta$ and $\mathcal{U}_n^{k_1,k_2}(\delta,h,c)$ as in the discussion just above Lemma A.5. Fix $C > 1$ and $\zeta > 0$. For $\delta \in (0,1)$, let $m_h^\delta := \lfloor (1-\delta)h \rfloor$ by as in Lemma 3.7. There exists $\delta_* > 0$ (depending only on C and ζ) such that for each $\delta \in (0,\delta_*]$, there exists $n_* = n_*(\delta, C, \zeta) \in \mathbb{N}$ such that the following is true.*

Suppose $n \geq n_*$, $(h,c) \in \left[C^{-1}n^{1/2}, Cn^{1/2}\right]_{\mathbb{Z}}$, and $(k_1,k_2) \in (-\infty, \Bbbk_\delta]_{\mathbb{Z}} \times [0, \Bbbk_\delta]_{\mathbb{Z}}$. Let x be a realization of $X_1 \dots X_{K^H_{n,m_h^\delta}}$ for which $(K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta}) \in \mathcal{U}_n^{k_1,k_2}(\delta,h,c)$ and $Q^H_{n,m_h^\delta} \geq \zeta n^{1/2}$. There is a constant $a_0 > 0$, depending only on C and ζ, such that

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{E}_n^{h,c} \,|\, X_1 \dots X_{K^H_{n,m_h^\delta}} = x,\, I > n \right) \\
&\quad \preceq \begin{cases} 2^{-3k_1} \exp\left(-a_0 2^{-2k_1 + 2k_2}\right) \delta^{-2} n^{-1}, & k_1 \geq 0 \\ \exp\left(-a_0 2^{-2k_1} - a_0 2^{-k_1+k_2}\right) \delta^{-2} n^{-1}, & k_1 < 0, \end{cases} \qquad (83)
\end{aligned}$$

with the implicit constant depending only on C and ζ.

Proof. Set $r_h^\delta := h - m_h^\delta$. Let $(h,c) \in \left[C^{-1}n^{1/2}, Cn^{1/2}\right]_{\mathbb{Z}}^2$. Let x be a realization of $X_1 \dots X_{K^H_{n,m_h^\delta}}$ as in the statement of the lemma, so $0 < K^H_{n,m_h^\delta} < I$, $(K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta}) \in \mathcal{U}_n^{k_1,k_2}(\delta,h,c)$, and $\rfloor(x) := \mathcal{N}_{C}(\mathcal{R}(x)) \geq \zeta n^{1/2}$. Also define $(J^H_{n,r_h^\delta}, L^H_{n,r_h^\delta})$ as in Section 3.1 and let $R_n(x)$ be as in (29).

By Lemma 2.11, we can find $A > 0$, depending only on p, such that for any such realization x we have

$$\mathbb{P}\left(\sup_{j \in [1, n-|x|-1]} |X(|x|+1, j)| \leq A(n-|x|)^{1/2} \,|\, R_n(x) \right) \geq \frac{1}{2}.$$

Since x is chosen so that $(K^H_{n,m^\delta_h}, Q^H_{n,m^\delta_h}) \in \mathcal{U}^{k_1,k_2}_n(\delta,h,c)$, we have $n - |x| \asymp 2^{2k_2}\delta^2 n$ (with universal implicit constant). By combining this with Lemma 2.6, we see that we can find a $\delta_* > 0$, depending only on C, such that for $\delta \in (0,\delta_*]$ we have

$$\begin{aligned}\mathbb{P}\left(R_n(x), \mathcal{N}_{\boxed{\mathrm{C}}}(X(|x|+1,n)) \leq \mathfrak{c}(x)\right) &\succeq (n-|x|)^{-1/2}\mathbb{P}\left(\sup_{j\in[1,n-|x|-1]}|X(|x|+1,j)| \leq \mathfrak{c}(x) \,|\, R_n(x)\right)\\ &\succeq (n-|x|)^{-1/2} \succeq 2^{-k_1}\delta^{-1}n^{-1/2} \end{aligned}\tag{84}$$

with the implicit constant depending only on C.

In the case $k_1 \geq 0$, Proposition 2.2 and Lemma 2.3 imply

$$\begin{aligned}\mathbb{P}\left((J^H_{n,r^\delta_h}, L^H_{n,r^\delta_h}) = (n-|x|-1, c-\mathfrak{c}(x))\right) &\preceq \left(2^{-4k_1}\exp\left(-a_0 2^{-2k_1+2k_2}\right) + o_{\delta h}(1)\right)\delta^{-3}h^{-3}\\ &\preceq \left(2^{-4k_1}\exp\left(-a_0 2^{-2k_1+2k_2}\right) + o_{\delta^2 n}(1)\right)\delta^{-3}n^{-3/2}\end{aligned}$$

with $a_0 > 0$ and the implicit constants depending only on C, and the $o_{\delta^2 n}(1)$ depending only on $\delta^2 n$ (and in particular not on x, k_1, or k_2). It follows that we can find $n_* = n_*(\delta, C)$ such that for each $(k_1,k_2) \in [0,\Bbbk_\delta]_{\mathbb{Z}} \times [0,\Bbbk_\delta]_{\mathbb{Z}}$, and each realization x as above, we have

$$\mathbb{P}\left((J^H_{n,r^\delta_h}, L^H_{n,r^\delta_h}) = (n-|x|-1, c-\mathfrak{c}(x))\right) \preceq 2^{-4k_1}\exp\left(-a_0 2^{-2k_1+2k_2}\right)\delta^{-3}n^{-3/2}\tag{85}$$

By combining this with (31) and (84), we obtain (83) in the case $k_1 \geq 0$.

In the case $k_1 < 0$, (23) of Lemma 2.8 implies

$$\mathbb{P}\left((J^H_{n,r^\delta_h}, L^H_{n,r^\delta_h}) = (n-|x|-1, c-\mathfrak{c}(x))\right) \preceq \exp\left(-a_0 2^{-2k_1} - a_0 2^{-k_1+k_2}\right)\delta^{-3}n^{-3/2}$$

for a constant $a_0 > 0$ depending only on C. By combining this with (84) we obtain (83) in the case $k_1 < 0$. ☐

Lemma A.8. *Let $C > 1$ and $q \in (0,1)$. For $(h,c) \in \mathbb{N}^2$ and $\delta > 0$, write $m^\delta_h = \lfloor (1-\delta)h\rfloor$ and $m^\delta_c = \lfloor(1-\delta)c\rfloor$. There exists $\delta_* > 0$ and $\zeta > 0$ (depending only on C and q) such that for each $\delta \in (0,\delta_*]$, we can find $n_* = n_*(\delta,\zeta,C,q) \in \mathbb{N}$ such that for $n \geq n_*$ and $(h,c) \in \left[C^{-1}n^{1/2}, Cn^{1/2}\right]_{\mathbb{Z}}$, we have*

$$\mathbb{P}\left(Q^H_{n,m^\delta_h} \wedge Q^C_{n,m^\delta_c} \geq \zeta n^{1/2} \,|\, \mathcal{E}^{h,c}_n\right) \geq 1-q.$$

Proof. Let $(h,c) \in \left[C^{-1}n^{1/2}, Cn^{1/2}\right]_{\mathbb{Z}}$. Observe that if $\delta \in (0,1/2)$, $Q^C_{n,m^\delta_c} < (2C)^{-1}n^{1/2}$, $I > n$, and $\mathcal{E}^{h,c}_n$ occurs, then $K^H_{n,m^\delta_h} \geq K^C_{n,m^\delta_c}$, and hence $Q^H_{n,m^\delta_h} \geq m^\delta_c \geq (2C)^{-1}n^{1/2}$. Indeed, this follows since, by definition,

$$\inf_{i\in[K^H_{n,m^\delta_h}+1,n]_{\mathbb{Z}}} \mathcal{N}_{\mathrm{H}}(X(1,i)) \geq m^\delta_h \geq (2C)^{-1}n^{1/2}$$

and

$$\inf_{i\in[K^C_{n,m^\delta_c}+1,n]_{\mathbb{Z}}} \mathcal{N}_{\mathrm{C}}(X(1,i)) \geq m^\delta_c.$$

Therefore, for $\zeta \leq (2C)^{-1}$,

$$\begin{aligned}&\mathbb{P}\left(\mathcal{E}^{h,c}_n,\, Q^C_{n,m^\delta_c} < \zeta n^{1/2} \,|\, I > n\right)\\ &\leq \mathbb{P}\left(\mathcal{E}^{h,c}_n,\, Q^H_{n,m^\delta_h} \geq (2C)^{-1}n^{1/2},\, F^H_{n,b}(\zeta) \,|\, I > n\right),\end{aligned}\tag{86}$$

where $F^H_{n,b}(\zeta)$ is the event of Lemma A.5 with $b = (2C)^{-1}$.

By Lemma A.7 (applied with $(2C)^{-1}$ in place of ζ), we can find $\delta_* > 0$, depending only on C, such that for each $\delta \in (0,\delta_*]$, there exists $\widetilde{n}_* = \widetilde{n}_*(\delta,C) \in \mathbb{N}$ such that for each $n \geq n_*$, each $(h,c) \in \left[C^{-1}n^{1/2}, Cn^{1/2}\right]_{\mathbb{Z}}$, and each $(k_1,k_2) \in (-\infty,\Bbbk_\delta]_{\mathbb{Z}} \times [0,\Bbbk_\delta]_{\mathbb{Z}}$, we have

$$\mathbb{P}\left(\mathcal{E}^{h,c}_n \,|\, E^{k_1,k_2}_n(\delta,h,c),\, I > n\right) \preceq \begin{cases} 2^{-3k_1}\exp\left(-a_0 2^{-2k_1+2k_2}\right)\delta^{-2}n^{-1}, & k_1 \geq 0\\ \exp\left(-a_0 2^{-2k_1} - a_0 2^{-k_1+k_2}\right)\delta^{-2}n^{-1}, & k_1 < 0,\end{cases}\tag{87}$$

with $a_0 > 0$ the implicit constants depending only on C, where here

$$E_n^{k_1,k_2}(\delta,h,c) := \left\{ (K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta}) \in \mathcal{U}_n^{k_1,k_2}(\delta,h,c),\, Q^H_{n,m_h^\delta} \geq (2C)^{-1} n^{1/2},\, F^H_{n,b}(\zeta) \right\}.$$

By the second assertion of Lemma A.5, for any given $\alpha \in (0,1)$, we can find $\zeta > 0$ (depending on C and α) such that for each $\delta \in (0,\delta_*]$, there exists $n_* = n_*(\delta,\zeta,C) \geq \widetilde{n}_*$ such that for each $n \geq n_*$, $(k_1,k_2) \in (-\infty, \Bbbk_\delta]_{\mathbb{Z}} \times [0,\Bbbk_\delta]_{\mathbb{Z}}$, and $(h,c) \in \left[C^{-1}n^{1/2}, Cn^{1/2}\right]_{\mathbb{Z}}$, we have

$$\mathbb{P}\left(E_n^{k_1,k_2}(\delta,h,c) \,|\, I > n \right) \preceq 2^{k_1 \vee 0 + k_2} \delta^2 \alpha$$

By (87), for $n \geq n_*$ and $(k_1,k_2) \in (-\infty, \Bbbk_\delta]_{\mathbb{Z}} \times [0,\Bbbk_\delta]_{\mathbb{Z}}$ we have

$$\mathbb{P}\left(\mathcal{E}_n^{h,c} \cap E_n^{k_1,k_2}(\delta,h,c) \,|\, I > n \right) \preceq \begin{cases} 2^{-2k_1+k_2} \exp\left(-a_0 2^{-2k_1+2k_2}\right) n^{-1}\alpha, & k_1 \geq 0 \\ 2^{k_2} \exp\left(-a_0 2^{-2k_1} - a_0 2^{-k_1+k_2}\right) n^{-1}\alpha, & k_1 < 0, \end{cases}$$

with the implicit constants depending only on C. By summing over all such k_1 and k_2, we infer

$$\mathbb{P}\left(\mathcal{E}_n^{h,c},\, Q^H_{n,m_h^\delta} \geq (2C)^{-1} n^{1/2},\, F^H_{n,b}(\zeta) \,|\, I > n \right) \preceq \alpha n^{-1}.$$

By combining this with (86) and Lemma A.6, we get

$$\mathbb{P}\left(Q^C_{n,m_c^\delta} < \zeta n^{1/2} \,|\, \mathcal{E}_n^{h,c} \right) \preceq \alpha,$$

with the implicit constant depending only on C. By symmetry and the union bound, also

$$\mathbb{P}\left(Q^C_{n,m_c^\delta} \wedge Q^H_{n,m_h^\delta} < \zeta n^{1/2} \,|\, \mathcal{E}_n^{h,c} \right) \preceq \alpha.$$

By choosing α sufficiently small (and hence ζ sufficiently small and n_* sufficiently large), depending only on q and C, we conclude. □

Proof of Proposition 3.7. Fix $\alpha > 0$ to be chosen later, depending only on C and q. Given $q \in (0,1)$, first choose $\delta_2 > 0$ sufficiently small that the conclusion of Lemma A.6 holds with δ_2 in place of δ_*. Then choose $\zeta > 0$ and $\delta_1 \in (0,\delta_1]$ such that the conclusion of Lemma A.8 holds with δ_1 in place of δ_* and α in place of q. Then choose $\delta_* \in (0,\delta_1]$ such that the conclusion of Lemma A.7 holds with this choice of ζ. Then for $\delta \in (0,\delta_*]$, there exists $\widetilde{n}_* = \widetilde{n}_*(\delta,C,\alpha)$ which simultaneously satisfies the conditions of Lemmas A.6, A.7, and A.8.

By the first assertion of Lemma A.5, we can find $n_* = n_*(\delta,C,\alpha) \geq \widetilde{n}_*$ such that for $n \geq n_*$ and $(k_1,k_2) \in (-\infty, \Bbbk_\delta]_{\mathbb{Z}} \times [0,\Bbbk_\delta]_{\mathbb{Z}}$, we have

$$\mathbb{P}\left((K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta}) \in \mathcal{U}_n^{k_1,k_2}(\delta,h,c) \,|\, I > n \right) \preceq 2^{k_1 \wedge 0 + k_2 \wedge 0} \delta^2,$$

with the implicit constants depending only on C. By combining this with Lemma A.7, we infer

$$\begin{aligned} &\mathbb{P}\left(\mathcal{E}_n^{h,c},\, (K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta}) \in \mathcal{U}_n^{k_1,k_2}(\delta,h,c),\, Q^H_{n,m_h^\delta} \geq \zeta n^{1/2} \,|\, I > n \right) \\ &\quad \preceq \begin{cases} 2^{-2k_1+k_2} \exp\left(-a_0 2^{-2k_1+2k_2}\right) n^{-1}, & k_1 \geq 0 \\ 2^{k_2} \exp\left(-a_0 2^{-2k_1} - a_0 2^{-k_1+k_2}\right) n^{-1}, & k_1 < 0, \end{cases} \end{aligned}$$

with $a_0 > 0$ and the implicit constants depending only on C and ζ. By summing over a suitable subset of $(k_1,k_2) \in \mathbb{N}$, we infer that there exists $A > 2$, depending only on C, α, and ζ, such that with $\mathcal{U}_n^\delta(A,h,c)$ as in the statement of the lemma, we have

$$\mathbb{P}\left(\mathcal{E}_n^{h,c},\, (K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta}) \notin \mathcal{U}_n^\delta(A,h,c),\, Q^H_{n,m_h^\delta} \geq \zeta n^{1/2} \,|\, I > n \right) \leq \alpha n^{-1}.$$

By combining this with our choice of ζ, we find

$$\mathbb{P}\left(\mathcal{E}_n^{h,c},\, (K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta}) \notin \mathcal{U}_n^\delta(A,h,c) \,|\, I > n \right) \preceq \alpha n^{-1},$$

with the implicit constant depending only on C. By Lemma A.6,

$$\mathbb{P}\left((K^H_{n,m_h^\delta}, Q^H_{n,m_h^\delta}) \notin \mathcal{U}_n^\delta(A,h,c) \,|\, I > n \right) \preceq \alpha,$$

with the implicit constant depending only on C. We now conclude by choosing α sufficiently small that α times this implicit constant is at most q. □

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